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The failure criteria and deformational moduli of granular rock

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THE FAILURE CRITERIA AND DEFORMATIONAL MODULI

OF GRANULAR ROCK

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ABSTRACT

Analytical solutions of stress for composite material are obtained by means of mathematical theory of elasticity, assuming spherical inclusions and uniform displacements of boundaries of representative elements. These solutions show that the failure criteria of composite materials are complicated functions of the elastic moduli of matrix, inclusion and composite, and the volume ratio of matrix and inclusion. Combining this theory with Griffith's theory gives a new criteria for brittle failure of granular rock. This theory appears to provide a nearly perfect model for granular rocks, inasmuch as: a) most assumptions used in other criteria are eliminated, b) most phenomena in failure of brittle rocks can be described theoretically, and c) it is the most logical so far.

A simple formula that relates the elastic moduli of inclusion and matrix to the effective moduli of the composite is also derived as a part of the thesis. Comparison with experimental data indicates that it approximates the value better than other approximation formulas.

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LIST OF SYMBOLS

a : Radius of an inclusion

d : Radius of a composite sphere or one half the length of a side of an element

f : V_1/V

m : Arbitrary multiplier

r : Radial distance

u_i : Displacements

w_o : Displacement in vertical direction at outer boundary of a composite element

A, B, C, D, E, F : Constants for stress solutions

E : Young's modulus

S : Strength

V : Volume

α : a/d

α_n : Stress function constants

β : E_2/E_1

γ : ν_2/ν_1

δ : S_2/S_1

$\epsilon_i, \epsilon_{ij}$: Strain and strain tensor

η : Porosity

r, θ, ψ : Spherical coordinate axes

μ : Modulus of rigidity

ν : Poisson's ratio

λ : Lame's constant

λ_i, θ_i : Constants

σ_i, σ_{ij}: Stress and stress tensor

Subscripts to A to G, S, V, μ and ν,

1 : denotes matrix

2 : denotes inclusion

no subscript denotes effective value for composite.

CHAPTER I

INTRODUCTION

Rock is a naturally occuring composite material. This composite nature has been neglected in the engineering field dealing with rock, mainly because it is not practical to consider and it is virtually impossible to analyse theoretically. Geological defects in rocks, like faults, joints, etc., have more influence on designing an engineering structure in rock. However, the mechanical and physical properties of rock are the most important factors in such designs, and have to be determined before considering the forementioned geologic defects.

The mechanical and physical properties of rock are usually determined in a laboratory with small specimens cut from drill cores. Despite the extreme precautions taken in making and handling specimens, and carefully followed "standardized" techniques and methods in testing, the test results have shown a vast descrepancy in the properties of intact rock. This large variance has been accepted as one of the natures of rock, and statistics were heavily relied on to determine the proper values.

Recently, as the knowledge of rock mechanics has advanced, some analytical study on the basic properties of rock has been made for the better understanding of the behavior of rock under loads. But the main attention has been focused on the effect of "micro" cracks in an attempt to utilize Griffith's crack theory in formulating a failure criteria for brittle material like rock.

In this investigation, rock is treated as a two-element composite material. The three-dimensional theory of elasticity is used in an attempt to develop a rational basis for the study of basic properties of rock for which the inclusions can be assumed to be in a spherical shape.

A. Purpose and scope of the thesis. Since most rocks (not minerals) are composite in nature, it is essential to treat them as composite materials. With the exception of a few sedimentary and metamorphic origins, the inclusions are generally granular shaped. The "granular rock" here is intended to refer to composite material in which the inclusion is of granular shape and the matrix and inclusion are both elastic and brittle, i.e., rock.

The basic purpose of this thesis is two-fold: 1) to determine the strength variation of granular rock due to matrix-inclusion ratio and void ratio, and the basic strength of matrix and inclusion, and 2) to find a suitable expression to relate the elastic properties of granular rock to the elastic properties of matrix and inclusion(s). The results will enhance the understanding and knowledge of the physical and mechanical behavior of rock that is very different than other engineering materials.

B. Approach used in the investigation. The over-all properties of granular rock are considered to be quasi-homogeneous and quasi-isotropic. This assumes that the inclusions are "perfectly-disorderly" distributed homogeneously (1) throughout the matrix. Thus, a unit element containing one inclusion represents the material with respect to the over-all elastic properties and the volume ratio of inclusion to matrix. The materials composing the matrix and inclusions are assumed to be isotropic, homogeneous and linearly elastic.

For the sake of theoretical analysis, the inclusions are assumed to be spherical. Chapter II is concerned with the stress analysis of an element with boundary conditions derived from reasonable assumptions. Chapter IV is concerned with determination of effective elastic moduli. These theories are compared with other existing theories and data available in Chapters III and IV.

C. Literature review. In general, a homogeneous material with cracks or voids can be classified as a special kind of composite, the rigidity of the inclusion (void, crack) being considered to be zero. The attempt to find relations between the elastic properties and the strength of elastic solids and the effect of composite nature on the strength is not new. The main purpose of such investigations in the field of earth sciences is to understand the failure mechanism and deformational behavior, and in the other sciences it is to obtain stronger and stiffer engineering materials.

Price (2, 3) attempted to derive a relationship between quartz content and the strength of sandstone and siltstone. His results show that the strength of rock increases as the quartz content increases.

Judd and Huber (4) and D'Andrea et al (5) observed a curvilinear relationship between compressive strength and the density of rocks.

Willard and McWilliams (6, 7) studied transgranular-intergranular fracture of granular rocks. They measured the distance increments of a fracture trace within grains and along grain boundaries in a thin-sectioned disc of charcoal gray granite. They concluded that transgranular defects are the predominant factor influencing the fracture of charcoal gray granite at low rates of loading.

Brady (8, 9, 10) studied the brittle fracture of rock in relation to the density of microcracks in the rock, assuming a uniform stress distribution throughout the material. He concluded that total failure takes place when the total microcrack density reaches a critical value. He also showed that the Griffith theory is not useful to the macroscopic failure of brittle material.

Morgenstern and Phukan (11, 12) experimentally determined the relationships between the strength and porosity and the porosity and relative compressibility of Bunter sandstone. They found that the porosity increases compressibility and decreases the strength almost linearly.

Ishai and Cohen (13) made an experimental study of yield strength of epoxy composites and investigated the effect of filler and cavity content on the yield strength.

Walsh and Brace (13, 14, 15, 16) investigated the effect of various shapes of cracks on the compressibility of rock and the effects of grain size on the fracture of rock, both theoretically and experimentally.

Huang (17) used Weibul's theory to determine the relationship of porosity to strength and to the elastic modulus of aluminum specimens.

Bortz and Nagao (18) found a good linear relationship between flexural strength and bulk density of commercial tar-bonded basic brick.

Brown and Mostaghel (19), Coble and Kingery (20), Hall (21), and others are concerned with reinforcing engineering materials with inclusions that are stronger than the matrix.

The amount of theoretical work has been far less than that of experimental work. Goodier (22) appears to be the first to derive solutions for spherical and cylindrical inclusions in infinite media. Edwards (23) obtained solutions for spheroidal inclusions and cavities, Eshelby (24) for ellipsoidal inclusions, Sternberg and Sadowsky (25) for two spherical cavities, and Wilson and Gorie (26) for an imbedded spherical inclusion in an infinite elastic solid. These theories have been applied to composite materials (27); however, they are not applicable to composites where the distance between inclusions are smaller than about three times their diameter.

More extensive work has been carried out by many investigators on the study of the physical rather than the aforementioned mechanical properties of composite materials in relation to the properties of matrix and inclusions. Einstein (28) is apparently the first (29) to attempt such work. He studied effective viscosity of a viscous fluid containing rigid spherical inclusions. Later, various combinations of rigid, viscous or elastic matrix, and viscous, rigid, elastic, plastic or void inclusions were studied by Taylor (30), Froehlich and Sack (31), MacKenzie (32), and Oldroyd (33). Eshelby (24) seems to be the first to use the model in which the inclusion and matrix are both elastic materials.

In all the studies mentioned here, the distance between two adjacent inclusions is assumed to be very large compared to the size of spherical inclusions, so that the interaction between inclusions can be neglected. Thus the theories are valid only when the inclusion to the matrix volume ratio is very small (about 3 per cent or less).

Smallwood (34), Guth (35), Mooney (36), Kerner (37), and Sato and Furukawa (38) modified Einstein's equation to use viscous composite of higher ratio of inclusion to matrix.

Only recently (1960) has attention been turned to elastic heterogeneous (high inclusion to matrix ratio) material. Paul (39) was the first (29, 40) to obtain the bounds for the elastic moduli of heterogeneous solids. The upper and lower bounds were obtained by using the minimum potential energy theorem and the theorem of least work, respectively, of the theory of elasticity. Although these bounds are theoretically exact, they are too far apart to provide a good estimate of the effective Young's modulus.

Hashin (41) obtained approximate bounds for two or more phase heterogeneous solids with spherical inclusions using the variational theorems. He assumed that the individual matrix part surrounding an inclusion is also a sphere concentric with the inclusion. Later Hashin and Shtrikman (42) derived similar expressions without making assumptions about phase geometry, but the bounds were still too wide in most cases to be practical.

The use of a single experimentally determined parameter, which is probably dependent on the ratio of Young's moduli of matrix and inclusion of two-phase solids, has been proposed by Wu (43). While this expression gives values of effective Young's modulus for any composite, the parameter itself must be determined by experiments.

Approximate formulas for determining the overall elastic moduli of a multi-phase material composed of contiguous inclusions were obtained by Budiansky (44), assuming that the grains of each phase are "more or less" spherical. His explicit formula for spherical inclusions shows that the modulus of matrix reaches that of inclusions when the volume ratio exceeds 50 per cent.

Greszczuk used assumptions similar to Paul's in an attempt to obtain an approximate expression for the average elastic moduli for elastic inclusion and bounds for the rigid inclusion of composite solids from an engineering viewpoint.

CHAPTER II

ANALYSIS OF STRESS

In this chapter, the theoretical solutions for the stresses in a unit element are obtained on the basis of the mathematical theory of elasticity. We assume: a) that the substances are homogeneous, isotropic and linearly elastic, b) infinitesimal strain, c) absence of body forces in the medium, and d) uniform temperature distribution.

The basic differential equations governing behavior of such elastic solids are known to be:

a) equation of equilibrium

$$\sigma_{ji,j} = 0 \qquad (1)$$

b) strain-displacement relations

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad (2)$$

c) stress-strain relations

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right) \qquad (3)$$

d) compatibility equations

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} = \epsilon_{ik,jl} + \epsilon_{jl,ik} \qquad (4)$$

The solutions of an elasticity problem must satisfy equations (1) through (4) and the boundary conditions. In general, there are four types of boundary conditions that are given to a problem, i.e.: a) the distribution of forces on the surface is prescribed, b) the distribution of displacements on the surface is prescribed, c) forces are prescribed over a portion of the boundary and surface, and d) components of surface forces and components of surface displacements are prescribed over the boundary.

Depending on the type of problem, boundary conditions, and number of dimensions considered, it is sometimes convenient to solve a problem when the governing equations are set entirely in terms of stresses or entirely in terms of displacements. In particular, if the displacement method is used, the stresses are uniquely defined by the stress-strain and strain-displacement relations so that the compatibility equations need not be used.

Substituting u_i into equation (1), we have

$$(\lambda+\mu)\, u_{j,ji} + \mu u_{j,ii} = 0$$

or

$$u_{j,ji} + (1-2\nu)\, u_{j,ii} = 0 \qquad (5)$$

The problem is now reduced to solving equation (5) with given boundary conditions.

A. Assumptions and boundary conditions. First, we assume that the representive unit element is a cube containing a spherical inclusion, and that the mass lies in a uniform uniaxial load field. In order to analyse stress conditions in this element some simplification of the geometry of the element and the assumptions are needed.

When the heterogeneous material undergoes changes in geometry due to external load, the individual element also changes its shape. We visualize a cubic element whose sides are either perpendicular or parallel to the direction of the load (or, we can cut an element in such a way that the sides will be parallel or perpendicular to the direction). We assume that the boundaries of the cubic element remain straight after the deformation takes place. Thus, if the load is uniform and uni-directional, the boundaries of the element will undergo constant displacements. This assumption is theoretically correct if the inclusions are distributed in cubical arrangement. It is also reasonably valid for composites with homogeneously distributed inclusions (45).

It is evident that the problem becomes much easier if we replace the outer boundaries with spherical ones. To find an exact boundary condition of the spherical surface that is replaceable with the constant displacement of the straight plane is impossible without knowing exact displacement functions between straight boundary and the inclusion. However, as will be shown later, it is reasonable to assume that the displacement of spherical boundary is the same as that in a homogeneous material with the effective elastic moduli. That is, when the upper boundary of a cube deforms uniformly by w_o, the displacements of spherical plane of radius d within the cube are:

$$u_r = \frac{w_o}{2}\left[(1-\nu)+(1+\nu)\cos 2\theta\right]$$
$$u_\theta = -\frac{w_o}{2}(1+\nu)\sin 2\theta \qquad (6)$$

Thus, the problem is reduced to solving a composite sphere with given boundary conditions equivalent to constant displacements. Other boundary conditions are that the displacements and the stresses across the boundary of the inclusion are the same for the inclusion and matrix, i. e.,

$$\left.\begin{aligned}(u_i)_1 &= (u_i)_2 \\ (\sigma_{rr}, \sigma_{r\theta})_1 &= (\sigma_{rr}, \sigma_{r\theta})_2\end{aligned}\right\} \text{ at } r=a \qquad (7)$$

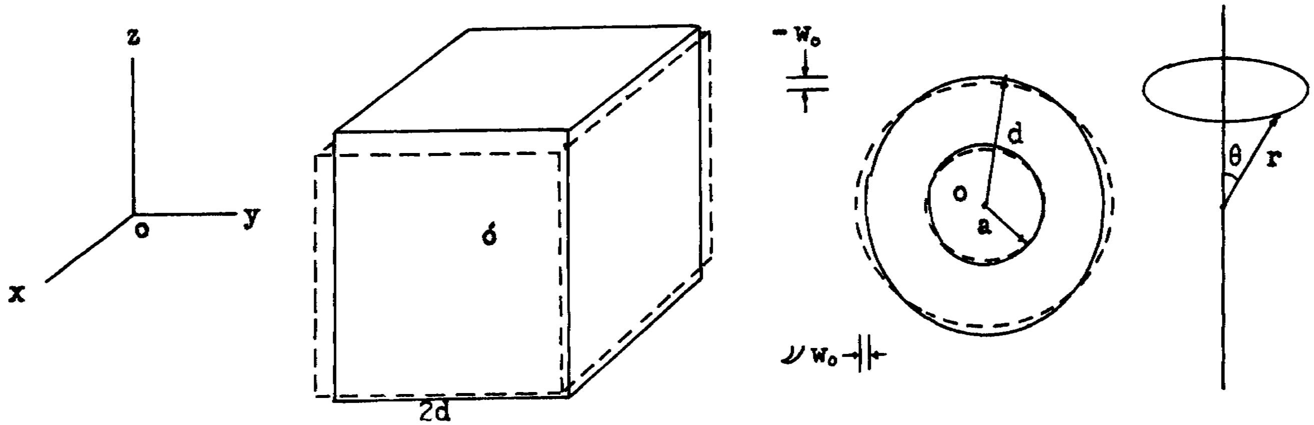


Figure 2-1. Replacement of boundary and coordinate system

B. Mathematical preliminary. The problem has been formulated in such a way that it can be solved in terms of series involving spherical harmonic functions with two variables. In this analysis, two types of solutions are used. They are, following Love's notation (66), type ω and type ϕ solutions. The ω type solution is given by

$$u_i = r^2 \omega_{n,i} + \alpha_n x_i \omega_n \tag{8}$$

where ω_n is a homogeneous solid harmonic of degree n. This satisfies the equation (5) provided that

$$\alpha_n = -2\,\frac{3n+1-2(2n+1)\nu}{n+5-4\nu}$$

The dilatation is

$$\Delta = \left[2n+(3+n)\right]\omega_n$$

The type solution is

$$u_i = \phi_{n,i} \tag{9}$$

where ϕ_n is any spherical solid harmonic of degree n. The dilatation vanishes for this solution.

Since the problem is axisymmetric, the solution is independent of angle ψ and we may use spherical coordinates with r and θ only. Changing the cartesian coordinates used in equations (8) and (9) into spherical coordinates, we have,

$$\begin{aligned} u_r &= r^2\frac{\partial\omega_n}{\partial r} + \alpha_n r\,\omega_n \\ u_\theta &= r\,\frac{\partial\omega_n}{\partial\theta} \end{aligned} \tag{10}$$

for ω type solutions, and

$$u_r = \frac{\partial \phi}{\partial r}$$
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{11}$$

for ϕ type solutions.

The formulas for the strains are:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$
$$\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$
$$\epsilon_{\psi\psi} = \Delta - \epsilon_{rr} - \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta \tag{12}$$
$$\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right), \quad \epsilon_{\theta\psi} = \epsilon_{\psi r} = 0$$

The general formulas for the stress-strain relations are:

$$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\psi\psi} = 2\mu\left[\frac{\nu}{1-2\nu}\Delta + (\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{\psi\psi})\right]$$
$$\sigma_{r\theta} = \mu\,\epsilon_{r\theta}, \quad \sigma_{\theta\psi} = \sigma_{\psi r} = 0 \tag{13}$$

Since the dilatation Δ vanishes for the ϕ type solutions, the stresses can be simply related to ϕ functions directly. By combining (11) and (12) and substituting results in equation (13), we have:

$$\sigma_{rr} = 2\mu\frac{\partial^2 \phi}{\partial r^2}$$
$$\sigma_{\theta\theta} = 2\mu\left(\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right)$$
$$\sigma_{\psi\psi} = -(\sigma_{rr} + \sigma_{\theta\theta}) \tag{14}$$
$$\sigma_{r\theta} = 2\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right), \quad \sigma_{\theta\psi} = \sigma_{\psi r} = 0$$

C. <u>Mathematical form of solutions.</u> The problem essentially is solving a Laplace equation in spherical form. When the solutions are independent of ψ, the general solution of this equation is known to be (46)

$$\omega, \phi = \sum_{n=0}^{\infty} r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} r^{-(n+1)} P_n(\cos\theta) \tag{15}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of nth degree in terms of $\cos\theta$. The terms $P_n(\cos\theta)$ are also called "zonal" harmonics, because the curves on the surface of a unit sphere along which such functions vanish are parallel to the equator of the surface, thus dividing the surface into zones. For a clear understanding of further development of stress solutions, a few zonal harmonics are set forth explicitly:

$$P_0 = 1 \qquad\qquad P_1 = \cos\theta$$

$$P_2 = \tfrac{1}{2}(3\cos^2\theta - 1) \qquad\qquad P = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

Solutions in terms of harmonics of positive degrees are used in problems relating to a body of finite size, and those of negative degrees in problems relating to a body with a small spherical cavity at the origin. We note that both sets of solutions are applicable to the matrix region. Since the problem is symmetric about z-axis and about the equator plane, it is easily seen that odd-numbered solutions may not be usable due to the term $\cos 2\theta$ in boundary conditions. With these facts considered, we choose following sets of equations:

a) ϕ_2 and ω_0 to account for the constant stress parts in region 1 (matrix) and region 2 (inclusion); ω_0 for hydrostatic stresses and ϕ_2 for non-hydrostatic stress parts;

b) ω_2 to satisfy the boundary conditions for region 2 and outer boundary conditions of region 1;

c) ω_{-3} to take care of the inner boundary of region 1;

d) ϕ_{-1} and ϕ_{-3} to meet the effect of the singularity in region 1; ϕ_{-1} for purely radial and ϕ_{-3} for pure shear part of stresses;

where subscripts of ω and ϕ refer to the power of r.

Multiplying each function with an arbitrary constant and superposing the resulting displacements and stresses, we find the following set of equations (16) for region 1:

$$u_r = \frac{-1}{r^2} A_1 - \frac{3}{4r^4} B_1 + \frac{r}{2} C_1 - \frac{2\lambda_1}{\lambda_2} rD_1 + \frac{3\nu_1}{\lambda_3} F_1 + \frac{\lambda_2}{4\lambda_1} \frac{1}{r^2} G_1$$

$$+ \left(- \frac{9}{4r^4} B_1 + \frac{3}{2} rC_1 + \frac{9\nu_1}{\lambda_3} r^3 F_1 + \frac{3\lambda_2}{4\lambda_1} \frac{1}{r^2} G_1 \right) \cos 2\theta$$

$$u_\theta = - \frac{3}{2} \left(\frac{1}{r^4} B_1 + rC_1 + r^3 F_1 + \frac{1}{r^2} G_1 \right) \sin 2\theta$$

$$\sigma_{rr} = \mu_1 \Big[\frac{4}{r^3} A_1 + \frac{6}{r^5} B_1 + C_1 - \frac{4\lambda_4}{\lambda_2} D_1 - \frac{3\nu_1}{\lambda_2} r^2 F_1 - \frac{\lambda_5}{\lambda_1} \frac{1}{r^3} G_1$$

$$+ \left(\frac{18}{r^5} B_1 + 3C_1 - \frac{9\nu_1}{\lambda_3} r^2 F_1 - \frac{3\lambda_5}{\lambda_1} \frac{1}{r^3} G_1 \right) \cos 2\theta \Big]$$

$$\sigma_{\theta\theta} = \mu_1 \Big[- \frac{2}{r^3} A_1 - \frac{6}{4r^5} B_1 + C_1 - \frac{4\lambda_5}{\lambda_2} D_1 - \frac{15\nu_1}{\lambda_3} r^2 F_1 + \frac{5}{2r^3} G_1$$

$$+ \left(- \frac{21}{2r^5} B_1 - 3C_1 - \frac{21\lambda_{10}}{\lambda_3} r^2 F_1 + \frac{3}{2r^3} G_1 \right) \cos 2\theta \Big]$$

$$\sigma_{\psi\psi} = \mu_1 \Big[- \frac{2}{r^3} A_1 - \frac{9}{2r^5} B_1 - 2C_1 - \frac{4\lambda_4}{\lambda_2} D_1 - \frac{3\lambda_{11}}{\lambda_3} r^2 F_1 - \frac{1}{2r^3} G_1$$

$$+ \left(- \frac{15}{2r^5} B_1 - \frac{3\lambda_6}{\lambda_3} r^2 F_1 + \frac{9}{2r^3} G_1 \right) \cos 2\theta \Big]$$

$$\sigma_{r\theta} = \mu_1 \left(\frac{12}{r^5} B_1 - 3C_1 - \frac{3\lambda_7}{\lambda_3} r^2 F_1 - \frac{3\lambda_4}{\lambda_1} \frac{1}{r^3} G_1 \right) \sin 2\theta$$

and for region 2, equations (17):

$$u_r = \frac{r}{2} A_2 - \frac{2\theta_1}{\theta_2} rB_2 + \frac{3\nu_2}{\theta_3} r^3 C_2 + \left(\frac{3}{2} rA_2 + \frac{9\nu_2}{\theta_3} r^3 C_2 \right) \cos 2\theta$$

$$u_\theta = - \frac{3}{2} \left(rA_2 + r^3 C_2 \right) \sin 2\theta$$

$$\sigma_{rr} = \mu_2 \Big[A_2 - \frac{4\theta_4}{\theta_2} B - \frac{3\nu_2}{\theta_3} r^2 C_2 + \left(3A_2 - \frac{9\nu_2}{\theta_3} r^2 C_2 \right) \cos 2\theta \Big]$$

$$\sigma_{\theta\theta} = \mu_2 \Big[A_2 - \frac{4\theta_4}{\theta_2} B - \frac{15\nu_2}{\theta_3} r^2 C_2 - \left(3A_2 + \frac{21\theta_{10}}{\theta_3} r^2 C_2 \right) \cos 2\theta \Big]$$

$$\sigma_{\psi\psi} = \mu_2 \Big[-2A_2 - \frac{4\theta_4}{\theta_2} B - \frac{3\theta_{11}}{\theta_3} r^2 C_2 - \frac{3\theta_6}{\theta_3} r^2 C_2 \cos 2\theta \Big]$$

$$\sigma_{r\theta} = -\, 3\mu_2 \left(A_2 + \frac{\theta_7}{\theta_3} r^2 C_2 \right) \sin 2\theta$$

where,

$\lambda_1 = 1 - 2\nu_1$ $\qquad$ $\theta_1 = 1 - 2\nu_2$

$\lambda_2 = 5 - 4\nu_1$ $\qquad$ $\theta_2 = 5 - 4\nu_2$

$\lambda_3 = 7 - 4\nu_1$ $\qquad$ $\theta_3 = 7 - 4\nu_2$

$\lambda_4 = 1 + \nu_1$ $\qquad$ $\theta_4 = 1 + \nu_2$

$\lambda_5 = 5 - \nu_1$ $\qquad$ $\theta_5 = 5 - \nu_2$

$\lambda_6 = 7 + 11\nu_1$ $\qquad$ $\theta_6 = 7 + 11\nu_2$

$\lambda_7 = 7 + 2\nu_1$ $\qquad$ $\theta_7 = 7 + 2\nu_2$

$\lambda_8 = 7 + 5\nu_1$ $\qquad$ $\theta_8 = 7 + 5\nu_2$

$\lambda_9 = 7 - 10\nu_1$ $\qquad$ $\theta_9 = 7 - 10\nu_2$

$\lambda_{10} = 2 - \nu_1$ $\qquad$ $\theta_{10} = 2 - \nu_2$

$\lambda_{11} = 7 + \nu_1$ $\qquad$ $\theta_{11} = 7 + \nu_2$

If the deformed cube was composed entirely of material 1 and elastic and homogeneous, the displacements and stresses on a spherical surface of radius a would be,

$$
\begin{aligned}
u_r &= \frac{w_o a}{2d} \left[(1 - \nu_1) + (1 + \nu_1) \cos 2\theta \right] \\
u_\theta &= -\frac{w_o a}{2d} (1 + \nu_1) \cos 2\theta \\
\sigma_{rr} &= \frac{E_1 w_o}{2d} (1 + \cos 2\theta) \\
\sigma_{r\theta} &= -\frac{E_1 w_o}{2d} \sin 2\theta \\
\sigma_{\theta\theta} &= \sigma_{\psi\psi} = 0
\end{aligned}
\qquad (18)
$$

The condition of uniform displacement, equation (18), is superposed to the equation (16) for region 1. Using boundary conditions (6) and (7), we obtain a set of nine equations from which the constants are determined. The three constants A_1, D_1, and B_1 are obtained independently from others.

$$A_1 = \frac{w_o(2\lambda_6+\lambda_1)(\lambda_1\theta_4\mu_2-\lambda_4\theta_1\mu_1)a^3 d^2}{3[(\theta_4\mu_2+2\theta_1\mu_1)\lambda_1 d^2-(\lambda_1\theta_4\mu_2-\lambda_4\theta_1\mu_1)a^3]}$$

$$B_1 = \frac{w_o\lambda_2}{6}\left[\frac{1}{d}-\frac{\lambda_1(2\lambda_6+\lambda_1)(\theta_4\mu_2+2\theta_1\mu_1)d^3}{(\theta_4\mu_2-2\theta_1\mu_1)\lambda_1 d^2-(\lambda_1\theta_4\mu_2-\lambda_4\theta_1\mu_1)a^3}\right]$$

$$C_1 = \frac{w_o\theta_2 d\ (2\lambda_6+\lambda_1)[\lambda_1\theta_4\mu_2-\lambda_4\theta_1\mu_1-\lambda_1^2\ (\theta_4\mu_2+2\theta_1\mu_1)d]}{6\theta_1[(\theta_4\mu_2+2\theta_1\mu_1)\lambda_1 d^2-(\lambda_1\theta_4\mu_2-\lambda_4\theta_1\mu_1)a^3]}$$

The remaining six constants may be obtained by solving the six simultaneous equations.

$$B_1 = Up/Lo$$

where,

$$Up = a^5 d^2 w_o \Bigg[\Big\{a^4 d^2 \langle d^3 X_1-\lambda_3\theta_3\theta_q(\mu_2-\mu_1)a^3\rangle(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)-3\lambda_3^2\theta_3\theta_q$$

$$*(\mu_2-\mu_1)(d^5 X_1-\theta_3 a^5 X_2)X_3\Big\}\langle\lambda_{12}a^2 d^2(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)-\lambda_3\lambda_4\theta_3\theta_q(\mu_2-\mu_1)$$

$$(3\lambda_3+2\lambda_1\lambda_q)X_3\rangle-\langle a^2 d(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)-\lambda_3\theta_3\theta_q(\mu_2-\mu_1)(3\lambda_3+2\lambda_1\lambda_q)X_3\rangle\Big\{$$

$$a^2(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)\langle\lambda_{12}d^3 X_1-\lambda_3\lambda_4\theta_3\theta_q(\mu_2-\mu_1)a^3\rangle-3\lambda_3^2\lambda_4\theta_3\theta_q(\mu_2-\mu_1)($$

$$d^2 X_1-\theta_3 a^5 X_2)X_3\Big\}\Bigg]$$

$$Lo = \Bigg[\Big\{a^7(5\lambda_1+3)(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)-(3\lambda_3+2\lambda_1\lambda_q)\Big\}\Big\{2\lambda_1 X_1+3\lambda_3\theta_3\langle\theta_q(\mu_2-$$

$$4\mu_1)+15\sigma_2\mu_2\rangle\Big\}\ X_3\Bigg]\Big\{a^4 d^2(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)\langle d^3 X_1-\lambda_3\theta_3\theta_q(\mu_2-\mu_1)a^3\rangle-$$

$$3\lambda_3^2\theta_3\theta_q(\mu_2-\mu_1)(d^5 X_1-\theta_3 a^5 X_2)X_3\Big\}-3d^4\langle a^2 d(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)-\lambda_3\theta_3\theta_q($$

$$\mu_2-\mu_1)(3\lambda_3+2\lambda_1\lambda_q)X_3\rangle\Bigg[d^3(2\lambda_1 X_1 X_4+3\theta_3 X_2 X_3)\Big\{a^2 X_1-\lambda_3\theta_3\langle\theta_q(\mu_2+\mu_1)+$$

$$15\sigma_2\mu_2\rangle d^2\Big\}-\lambda_3(d^5 X_1-\theta_3 a^5 X_2)\Big\{5\lambda_1 X_1+3\lambda_3\theta_3\langle\theta_q(\mu_2-4\mu_1)+15\sigma_2\mu_2\rangle X_3\Big\}\Bigg]$$

$$C_1 = w_o \frac{a^2(2\lambda_1 X_1 X_4 + 3\theta_3 X_2 X_3)[\lambda_{12} d^3 X_1 - \lambda_3\lambda_4\theta_3\theta_q(\mu_1-\mu_2)a^3] - 3\lambda_3^2\lambda_4\theta_3\theta_q(\mu_1-\mu_2)(d^5 X_1 - \theta_3 a^5 X_2)X_3}{3da^2 \; d^3 X_1 - \lambda_3\theta_3\theta_q(\mu_2-\mu_1)a^3 \; -9d\lambda_3^2\theta_3\theta_q(\mu_2-\mu_1)(d^5 X_1 - \theta_3 a^5 X_2)X_3}$$

$$- \frac{d^3(2\lambda_1 X_1 X_4 + 3\theta_3 X_2 X_3)\{a^2 X_1 - \lambda_3\theta_3[\theta_q(\mu_2-4\mu_1)+15\sigma_2\mu_2]d^2\} - \lambda_3(d^5 X_1 - \theta_3 X_2 a^5)\{5\lambda_1 X_1 + 3\lambda_3\theta_3[\theta_q(\mu_2-4\mu_1) + 15\sigma_2\mu_2]\}X_3}{a^5\{a^2(2\lambda_1 X_1 X_4 + 3\theta_3 X_2 X_3)[d^3 X_1 - \lambda_3\theta_3\theta_q(\mu_2-\mu_1)a^3] - 3\lambda_3^2\theta_3\theta_q(\mu_2-\mu_1)(d^5 X_1 - \theta_3 a^5 X_2)X_3\}} B_1$$

$$F_1 = \left[\lambda_3^2\theta_3\theta_q X_3(\mu_1-\mu_2)a^5(w_o\lambda_4 + 3dC_1) - X_3 d\left\{5\lambda_1 X_1 + 3\lambda_3\theta_3\langle\theta_q(\mu_2-4\mu_1)+15\sigma_2\mu_2\rangle\right\}B_1\right] / \; a^7 d(2\lambda_1 X_1 X_4 + 3\theta_3 X_2 X_3)$$

$$G_1 = \left[\lambda_3\theta_3\theta_q a^5(\mu_1-\mu_2)(w_o\lambda_4 - 3C_1 d) - 3\theta_3 d(a^7 X_2 F_1 + \lambda_3 X_5 B_1)\right] / \; 3X_1 a^2 d$$

$$A_2 = \frac{w_o\lambda_4}{3d} + \frac{\theta_7\mu_2 + 4\theta_3\mu_1}{(\theta_7-\theta_3)\mu_2 a^5}B_1 + C_1 + \frac{\lambda_3\theta_7\mu_2 - \lambda_7\theta_3\mu_1}{(\theta_7-\theta_3)\lambda_3\mu_2}a^2 F_1 + \frac{\lambda_1\theta_7\mu_1 - \lambda_4\theta_3\mu_1}{\lambda_1(\theta_7-\theta_3)\mu_2 a^3}G_1$$

$$C_2 = \frac{w_o\lambda_4}{3a^2 d} + \frac{B_1}{a^7} + \frac{C_1}{a^2} + F_1 + \frac{G_1}{a^5} - \frac{A_2}{a^2}$$

where,

$$X_1 = \theta_3\theta_q(\lambda_1\mu_2 - \lambda_4\mu_1) - 9\sigma_2\theta_2\mu_2$$

$$X_2 = 6\sigma_2\lambda_q\mu_2 + \theta_q(\lambda_3\mu_2 - \lambda_7\mu_1)$$

$$X_3 = \theta_8\mu_2 + 4\theta_q\mu_1$$

$$X_4 = \lambda_q\theta_8\mu_2 - \lambda_8\theta_q\mu_1$$

$$X_5 = \theta_q(\mu_2 - 4\mu_1) + 15\sigma_2\mu_2$$

$$\lambda_{12} = \sigma - \sigma_1$$

The calculation of constants, thus the stresses, is virtually impossible to do by hand. Therefore it is necessary to use a digital computer. The variations of constants with α (= a/d), β (= E_2/E_1), and γ (= ν_2/ν_1) are shown in Figures 2 through 6, for particular cases. It is seen that all constants, except A_2 and B_2, converge to zero as β becomes zero or α approaches one. This means that when α and β become such values, i.e., when the element is homogeneous, the stresses are constant and the displacements are in linear relationship with the radial distance. Computation by digital computer proved that the boundary conditions are also satisfied. This assures that the foregoing solutions are correct. It is interesting to note that the values change almost linearly with respect to γ (Figures 5 and 6).

D. Discussion on boundary conditions. In obtaining the theoretical solutions, it was assumed that the outer boundary of the spherical composite element will displace as if it were of a homogeneous material that has the properties of aver-all composite (equation (6)). Clearly, this is true when $\gamma = 1$, in view of the fact that the ν is the only factor governing the boundary equations, For $\gamma \neq 1$, however, it is not known if the assumption is correct.

As will be discussed later (Chapter IV) in detail, the value of effective Poisson's ratio lies between values of ν_1 and ν_2. The rigorous analysis does not give exact values of ν but only bounds of ν, which lie in the range between ν_1 and ν_2.

Using the average value of these bounds for ν may be practical when the gap between the bounds is narrow, but it may be very

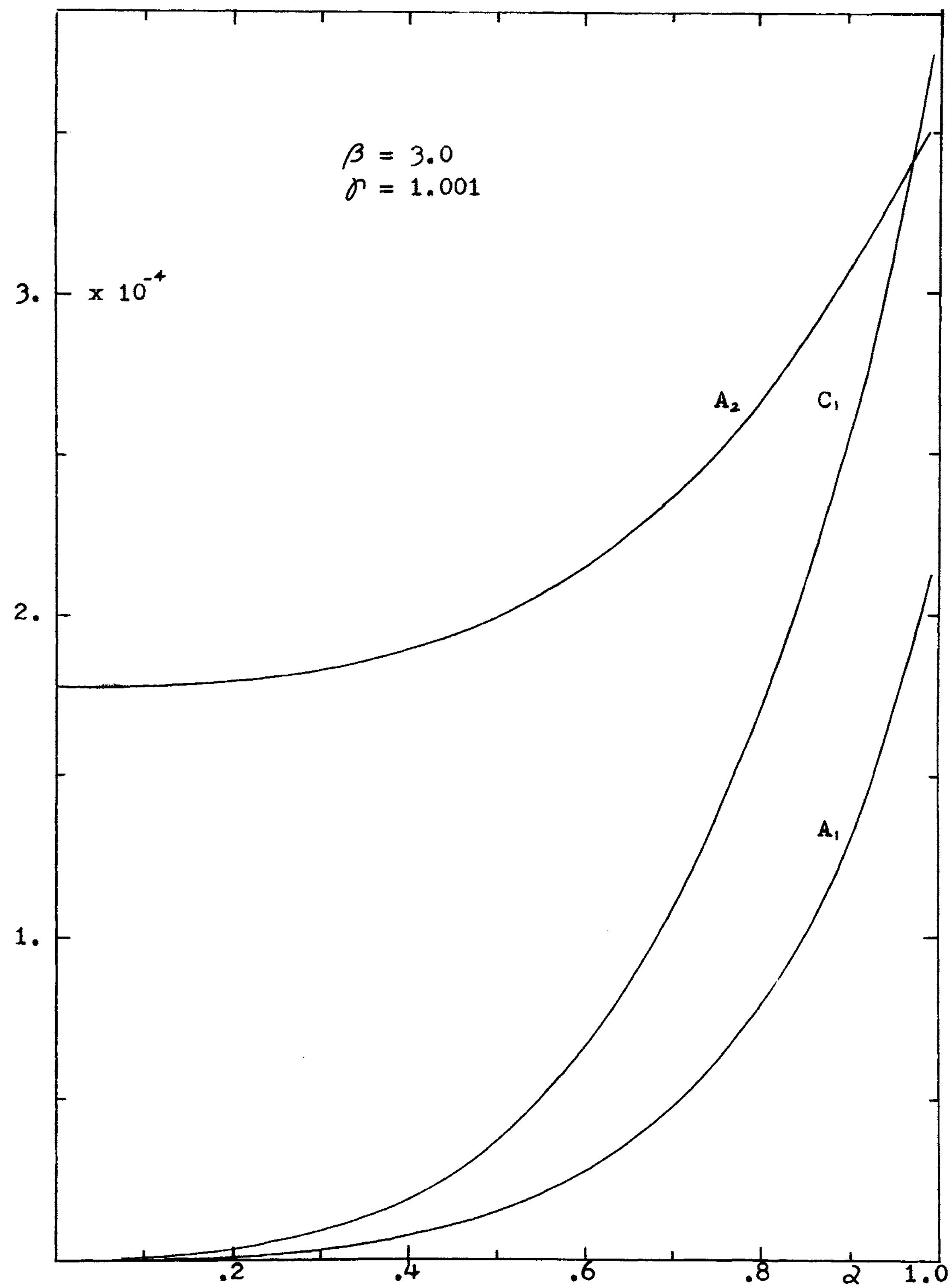


Figure 2-2. Constants vs. α

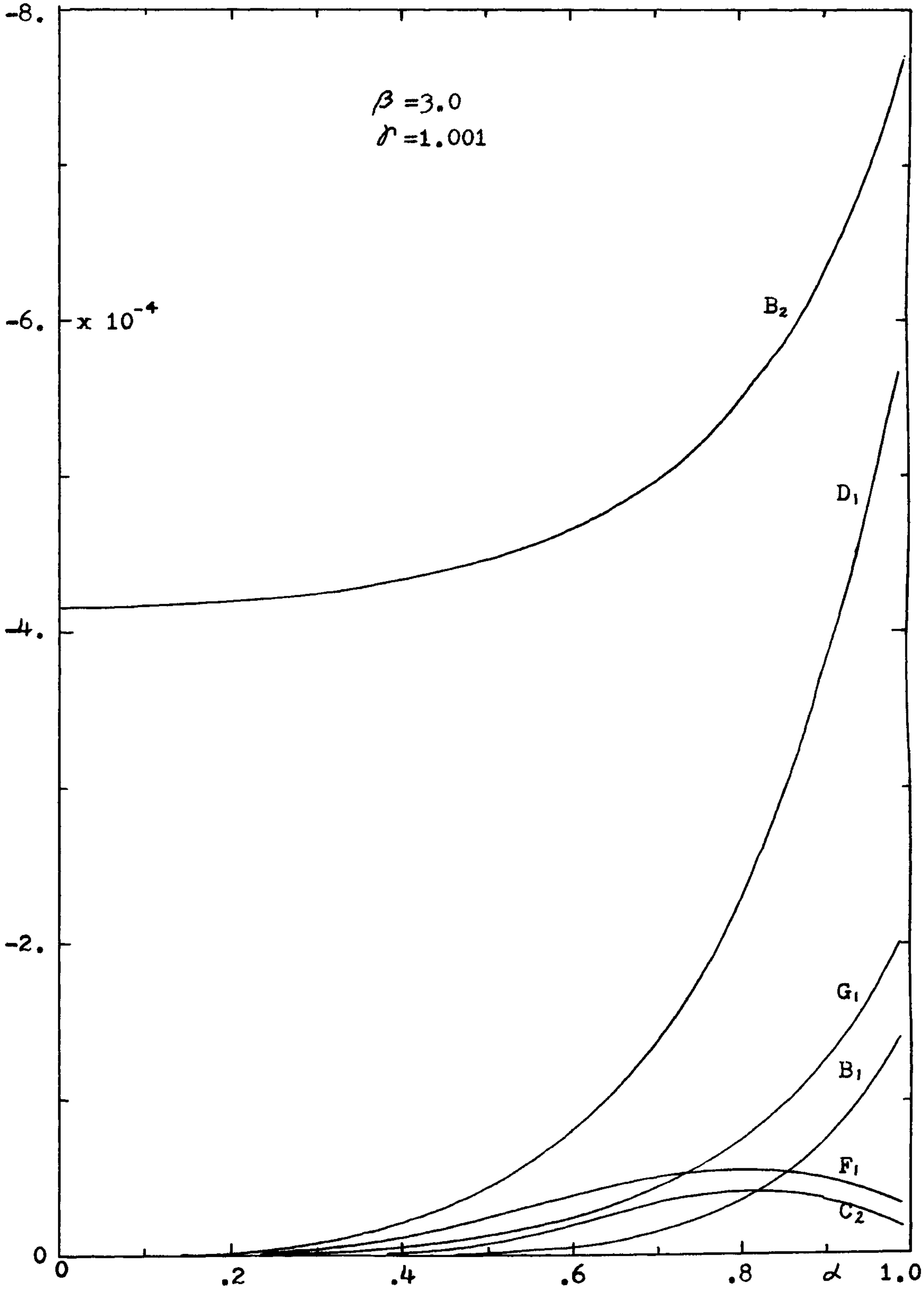


Figure 2-3. Values of constants vs. α

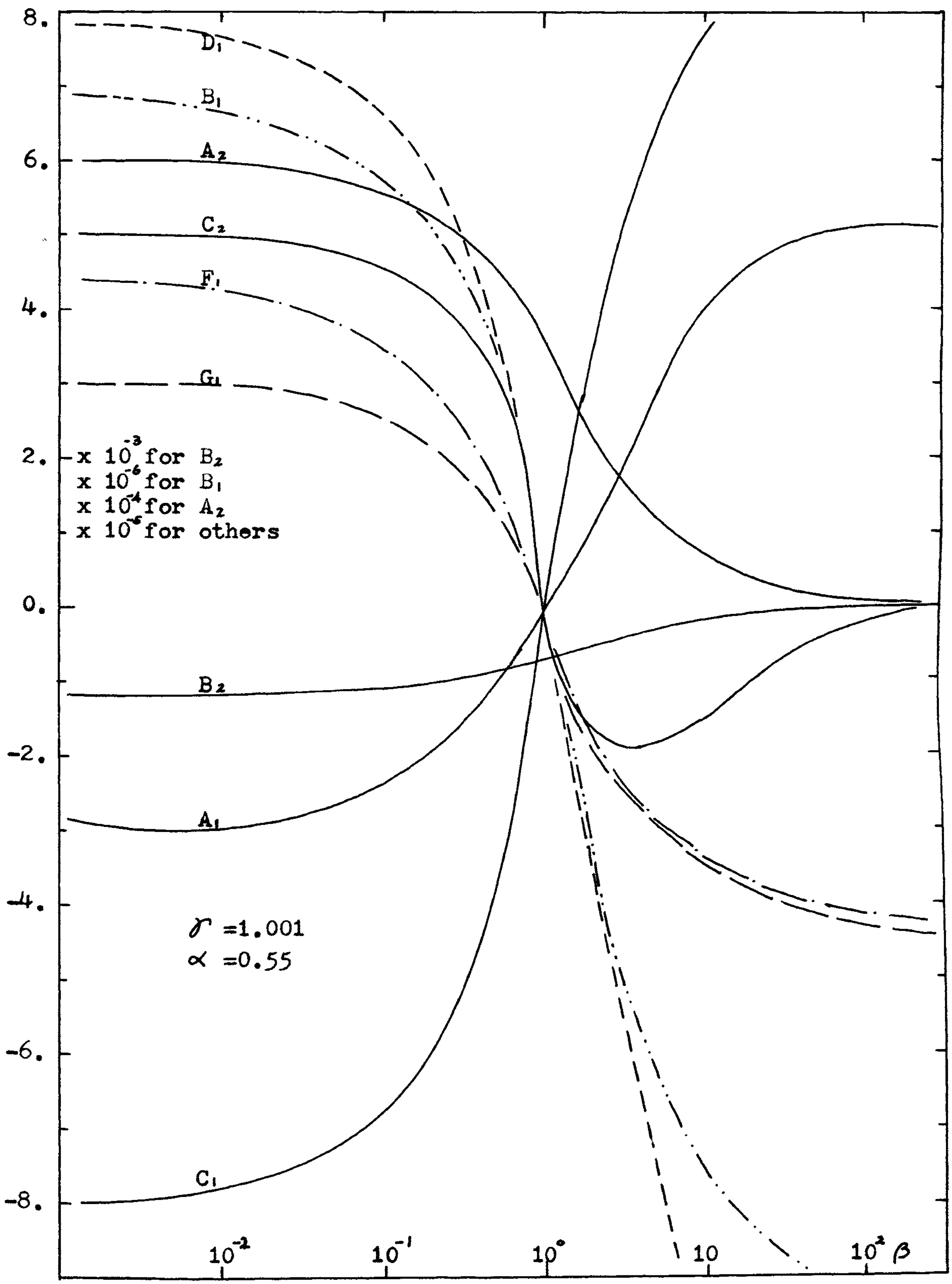


Figure 2-4. Constants vs. β

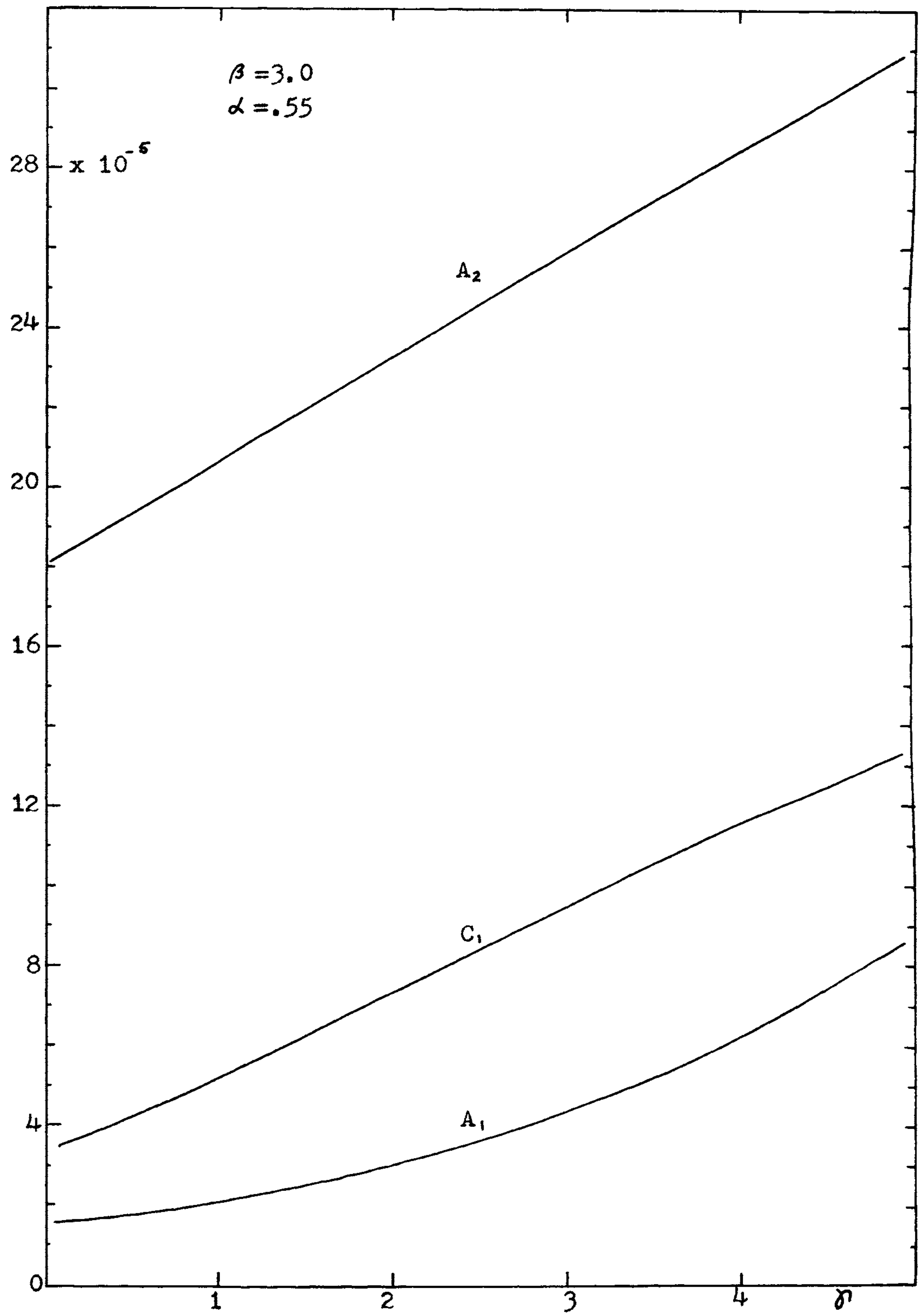


Figure 2-5. Constants vs. γ

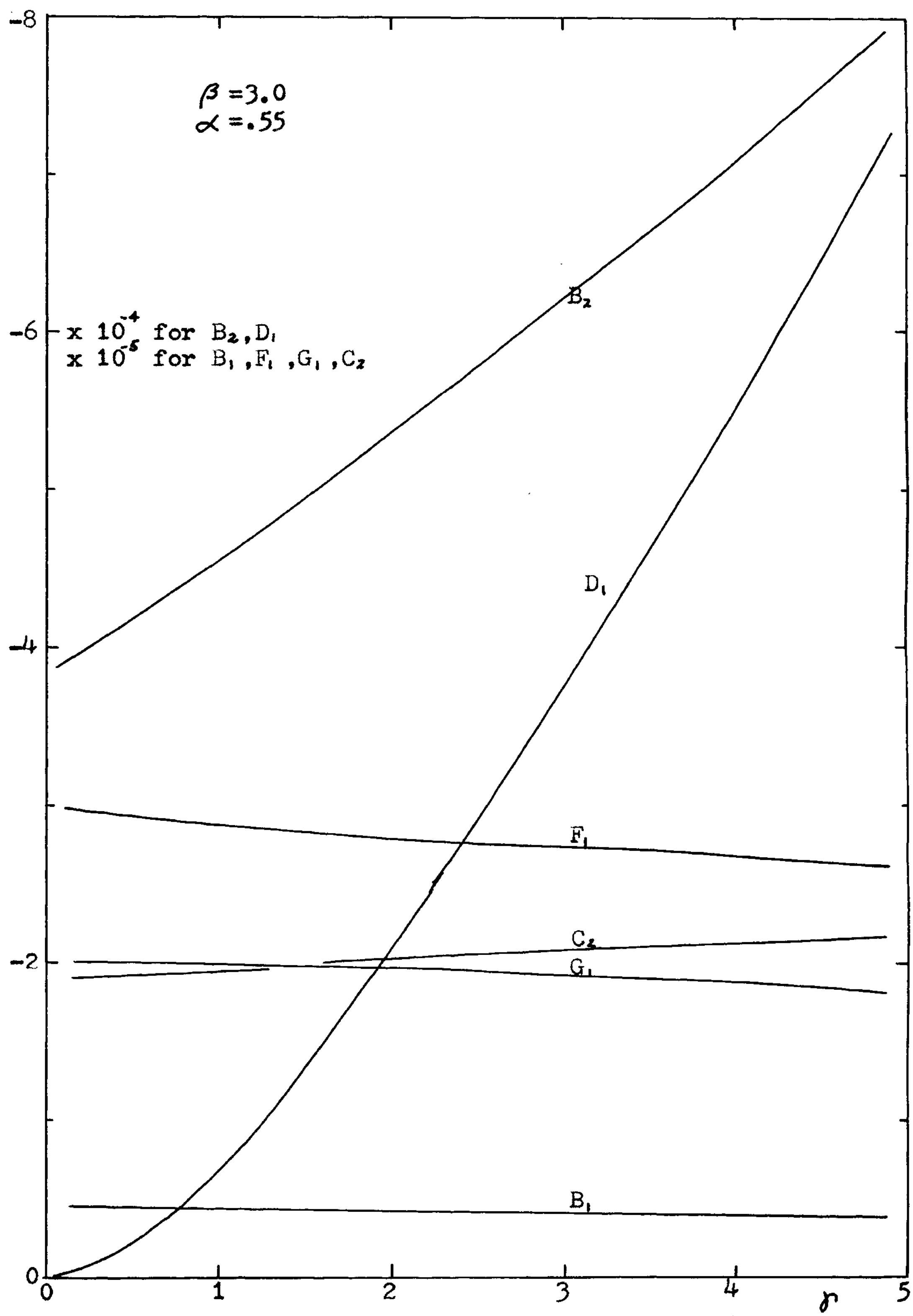


Figure 2-6. Constants vs. γ

incorrect when the gap is wide. It is conceivable, however, that ν_1 will have more effect than ν_2 in actual displacement, as the boundary lies within material 1. It was found through comparison with finite element solutions that the effective ν prescribes the actual boundary conditions very closely (Appendix B). The solutions describe the conditions for porous rock when β approaches zero.

The comparisons of stresses with finite element solutions for $\alpha = 5/8$, with different β and γ's are shown in Appendix A. All theoretical values seem to agree very well with those of finite element solutions, although the boundary conditions are somewhat different. It should be remembered that the finite element solutions are very rough approximations because of the small number of elements used in analysis, especially along the boundary (45).

E. Results of stress solutions. As expected, stress concentrations exist along the boundaries of inclusions. This affirms the common belief that the grain boundaries in rock are planes of weakness. Figures 7 through 10 show some results from the stress solutions for matrix region. The largest maximum principal stresses (max. σ_1) shown, tension being positive, are that occur at the grain boundaries due to the effective unit stress σ_0. Figure 10 shows the ratio of the maximum compressive stress (σ_c) on the grain boundary to the maximum tensile stress (σ_t) also on the grain boundary.

Whether the material breaks by maximum stress, maximum shear stress, or maximum extension stress, may depend on the type of material. But it is clear that such failure will always be initiated at the grain boundaries. The three principal stresses, maximum stress difference, and induced stresses in principal directions for several different values of α, β, and γ are listed in Appendix B.

The solutions also indicate that the principal stresses at points just inside and just outside the grain boundaries are different. This may explain why the grains break in some rocks under load even when the load is not sufficiently great to cause the failure in matrix region.

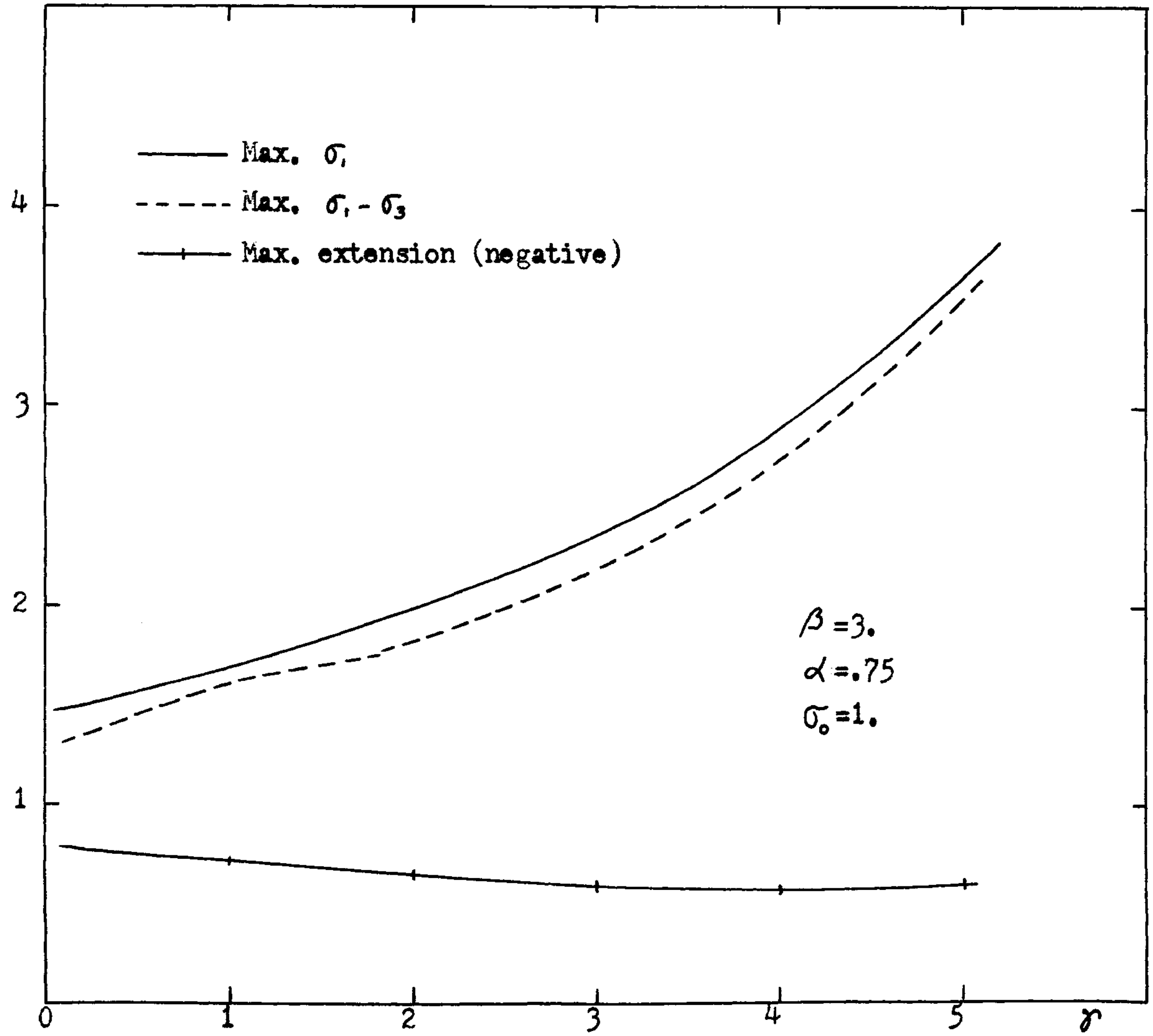


Figure 2-7. Maximum stress concentration factors vs. γ

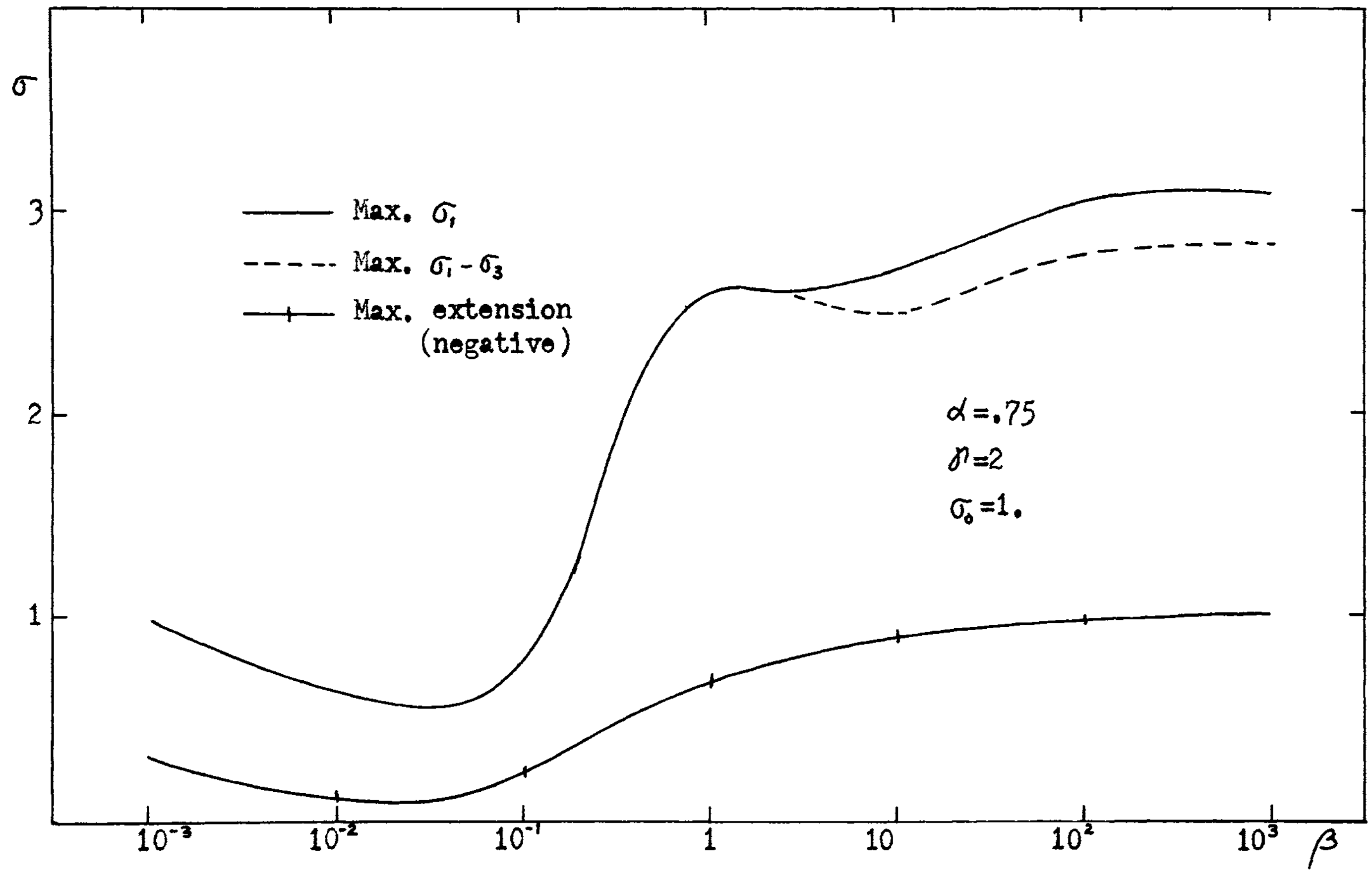


Figure 2-8. Maximum stress concentration factors vs. β

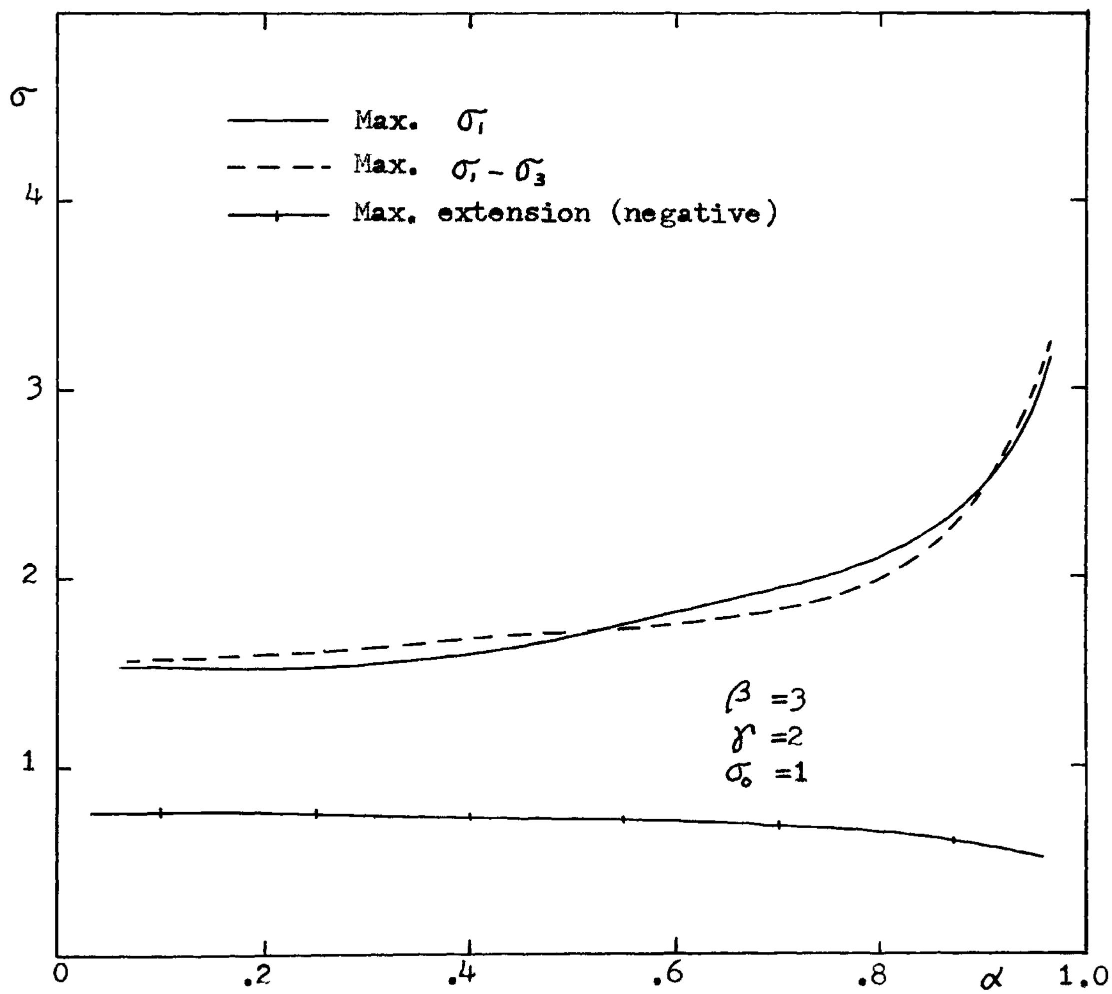


Figure 2-9. Maximum stress concentration factors vs. α

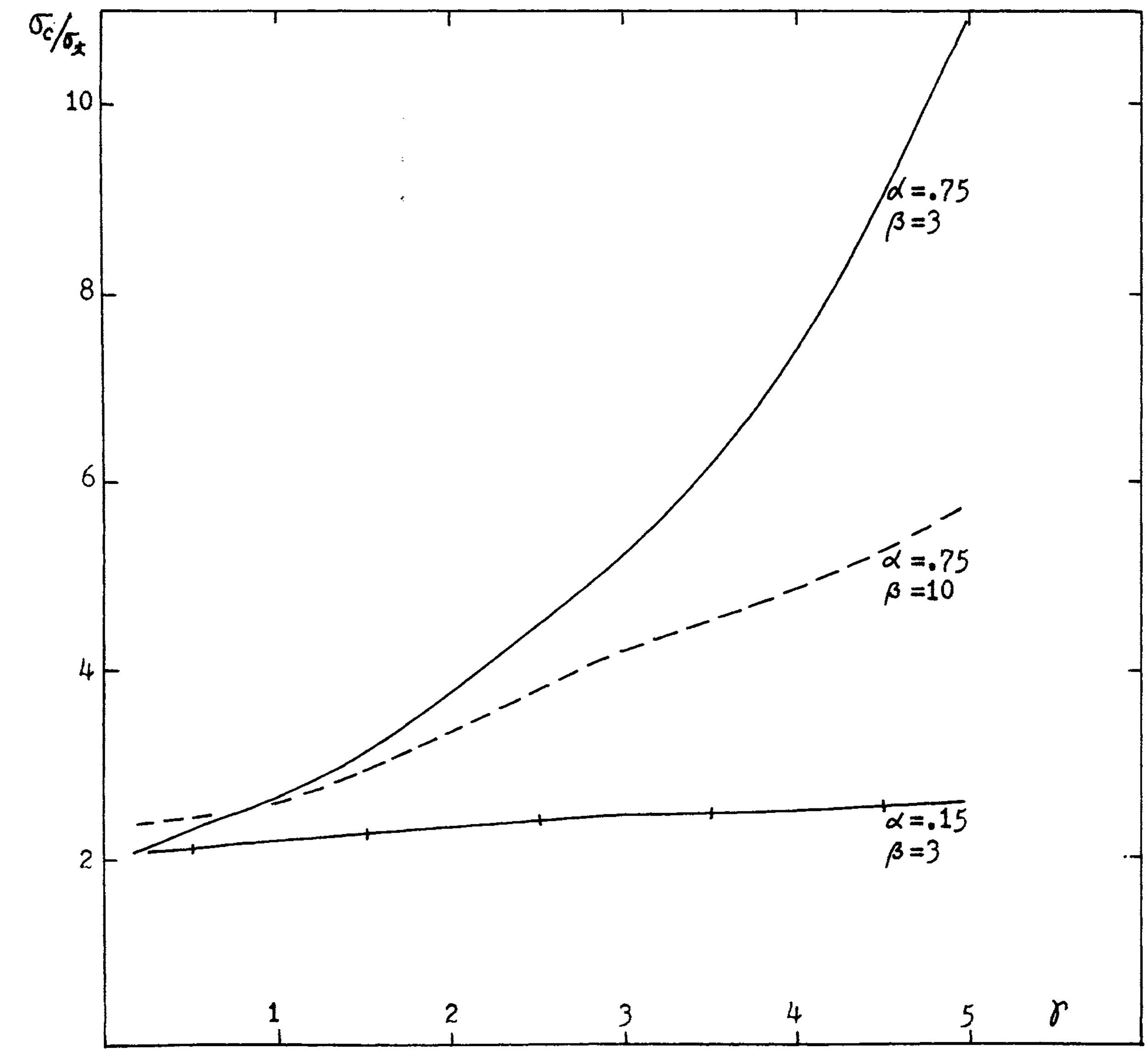


Figure 2-10. Ratio of maximum compressive to tensile stress vs. γ

CHAPTER III

FAILURE CRITERIA OF GRANULAR ROCK

There are many theories describing the conditions of failure of materials under mechanical load. For example, Jaeger (54), lists eight such theories. Most of these theories were developed for ductile materials such as metals and plastics. It should be pointed out that a failure criterion does not necessarily describe the actual failure mechanism, but that it only describes the condition under which failure may take place most of the time.

Among the proposed theories, only Coulomb-Navier and Mohr's theories have been found to be "reasonably valid" for rock on the basis of experimental studies (47). The Griffith's theory of brittle failure has become quite popular in the rock mechanics field in recent years. The main reason for this is that it is the first theory that provides a model for brittle material. The theory was developed for glass, based on the stress concentration around a micro crack, and has been proven to be valid for isotropic materials such as glass (48). Since all rocks have cracks (16, 49) to varying degrees, the Griffith theory should be applicable to rocks. However, direct application of this theory to rock has not been very successful (16, 49). A more obvious defect (51) of the theory is that it indicates that the ratio of uniaxial compressive strength to tensile strength is exactly eight, whereas it is well known that the ratio sometimes exceeds 100 in rocks. Otherwise, it gives a semiquantitative description of many common phenomena of rock (49). Hence, many attempts were made to remove this "defect" through modifications, but it may be that the defect was in the application, not in the original theory.

McClintock and Walsh (52) hypothesized that the cracks close up under pressure, developing friction on the friction surfaces, and Paul and Gangal (51) developed an idea of fracture hardening. However, experimental studies show that the coefficient of friction at the crack surface has to be very high (up to twice of actual value) in order that the compressive strengths exceed ten times the tensile strength (53). The hardening of fracture is very doubtful in brittle materials, especially at a relatively low pressure of uniaxial compression.

Again we must remember that the original Griffith theory was developed for glass, a homogeneous and isotropic material. Despite the fundamental fact that most rocks are notably inhomogeneous, all aforementioned theory neglected this property of rock. In fact, no study of effect of inhomogeneity under stress conditions is known to have been made to date (49).

The solutions from previous chapters provide a failure criteria for composites of ideally isotropic materials, i.e., for heterogeneous material without micro cracks. If we combine the Griffith theory and the effect of heterogeneity, it will describe the failure of granular rock better than any single existing theory.

It is proposed here that the Griffith characteristics be taken as intrinsic properties of each individual material composing the inclusion and matrix, and that the new failure criterion based on the effect of heterogeneity be used for rocks, especially for granular rocks. This new criterion is extremely difficult to put into a mathematical form. However, the concept is quite simple and may be expressed as

$$\sigma_f = \sigma\,(\sigma_0,\ E_1,\ \nu_1,\ S_1,\ \alpha,\ \beta,\ \gamma,\ \delta,\ G(r))$$

where S_1 is the strength of the matrix, δ is the strength ratio between the inclusion and matrix, and G(r) is the stress change due to Griffith cracks.

This criteria gives explanations to many questions heretofore unanswered.

a) The ratio of compressive and tensile strength of rock depends on the properties and their ratios of individual minerals composing the rock. For instance, the ratio may be exactly eight for individual minerals, but because of the heterogeneity, for rock with $\beta = 3$, $\gamma = 2$, and $\alpha = .75$, the stress ratio is about 5, which raises the strength ratio of the particular rock to 40. Thus the strength ratio of uniaxial loadings is theoretically explained.

b) The frequent failures along the grain boundaries have lead to the assumption that the micro cracks concentrate along such boundaries. According to the new criterion, high stress concentrations along the boundary, given by the stress solutions, are really responsible for such breakages.

c) This theory is capable of predicting whether the type of failure of a particular rock will be intergranular or intragranular. The principal stresses (maximum), σ_λ, are different for inside and outside the grain boundary. Therefore, when the strength ratio is exceeded by the ratio of stress

$(\sigma_1)_{inside}/(\sigma_1)_{outside}$, the failure will be transgranular, and vice versa.

It is well-known fact that rock is weaker when wet. Jaeger (54) suggested the use of effective stress for saturated rocks, i.e., to use $\sigma - p$ instead of σ, where p is pore water pressure. But the experimental results did not confirm this very well, especially for rocks with small porosities (49). The stress solutions indicate that the stress concentration factors are very high when the inclusion is liquid. The stresses for water inclusions are obtained by setting $\nu_2 = .5$ and $E_2 = 300{,}000$ psi (55). For example, for $\beta = .10$, $\nu_1 = .1$, and $\alpha = .95$, the maximum concentration factor is about 23.0, whereas it is about 8.5 when $\beta = .001$, which approximates dry porous material. Thus the proposed theory also gives the most logical explanation to failure of saturated rocks. These numbers are based on the assumption that the pore water is completely confined within each pore. Although a direct application may not be possible because of seepage of water in real rock, the theoretical values can be inferred to such problems.

One disadvantage in using this theory is the complexity of the expressions. Despite this, the theory provides a near perfect model for brittle, granular rocks in view of the facts that: a) it eliminates most assumptions used in other theories, b) it makes it possible to explain most phenomena that were not possible with other theories, and c) it is the most logical from the mathematical point of view.

A direct application here may not be feasible as in the case with the Griffith theory, because the theory gives only the conditions for "initial" failures. Some cracks due to initial failure may not lead to a complete failure, depending on the properties of the material.

Figure 1 shows the variation of the strength of porous rock with respect to its porosity for $\nu_1 = .3$. The ratios of compressive strengths to that at zero porosity are reciprocals of the ratio of maximum tensile stress to that for $\alpha = .15$, assuming here that the compressive failure is caused by these tensile stresses. When a dry porous rock is compressed (extended), the maximum compressive (tensile) stress occurs outside the outer spherical boundary, because the vertical stiffness is higher near the edges of the elemental cube. Thus the absolute maximum stresses for void inclusions cannot be found with the theoretical solutions. The ordinate on the tensile strength curve is the ratio of maximum tensile stress at $\theta = 90$ degrees to that occuring anywhere in the matrix, when the element is in tension.

Figure 2 shows the theoretical compressive to tensile strength ratio with respect to porosity. This was obtained by multiplying eight (from Griffith theory) to the ratio of maximum tensile stress developed in compression to the apparent stress. The apparent stress, σ, was calculated by multiplying effective (apparent) strain and effective Young's modulus, i.e., $\sigma = E\, w_o/d$. Porosity (η) can be calculated from $\eta - \alpha$ relation $\eta = \pi \alpha^3/6$.

Figure 3 shows the results from the Brazilian test (indirect tensile test) on pressure-sintered NiO, taken from reference 75. In plotting the experimental values, the original strength (i.e., tensile strength when $\eta = 0$) was assumed to be 22,000 psi.

Results of the unconfined (uniaxial) compression test with Bunter sandstone are shown in Figure 4, with the theoretical curve for

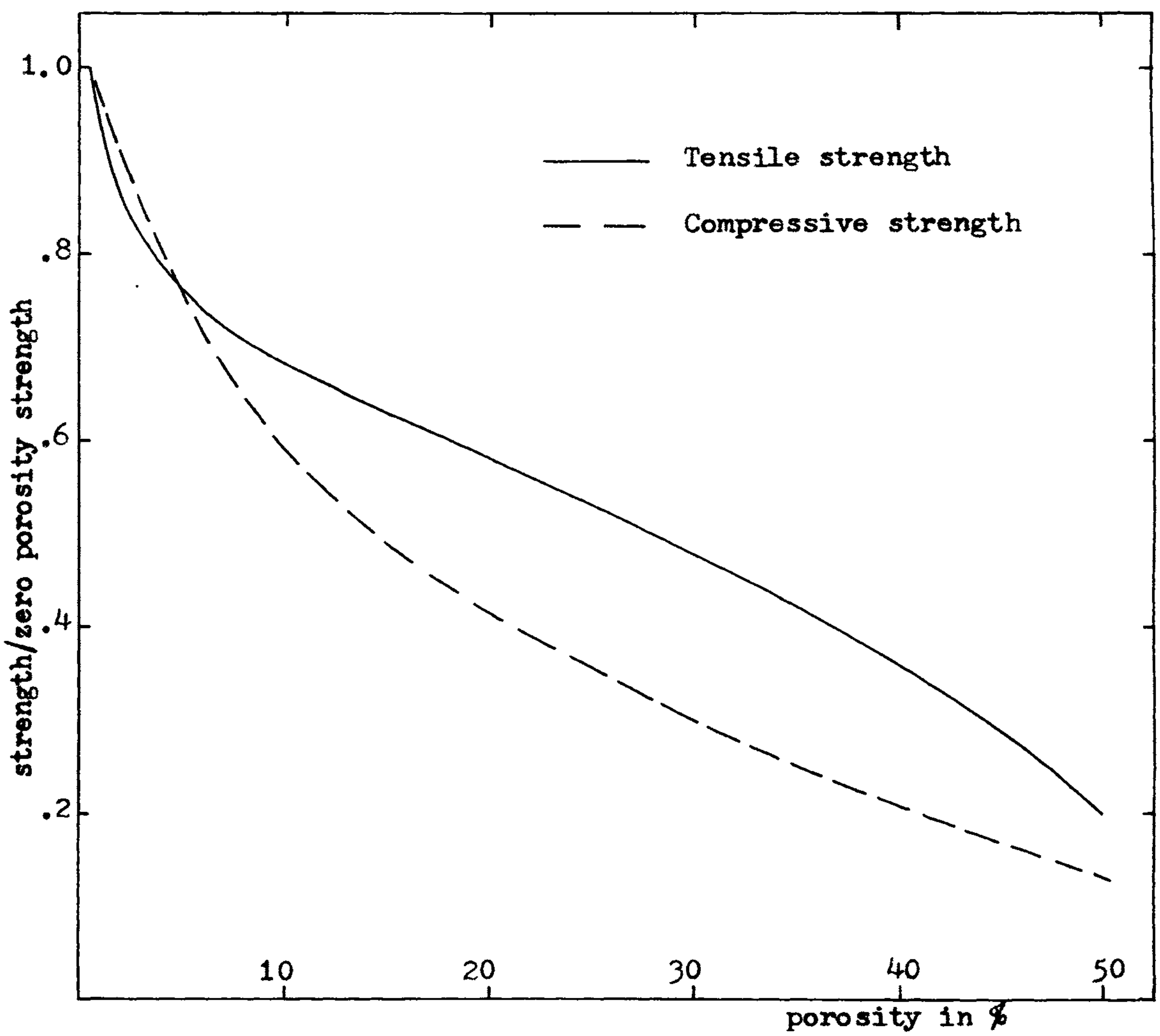


Figure 3-1. Variation of theoretical strengths with porosity

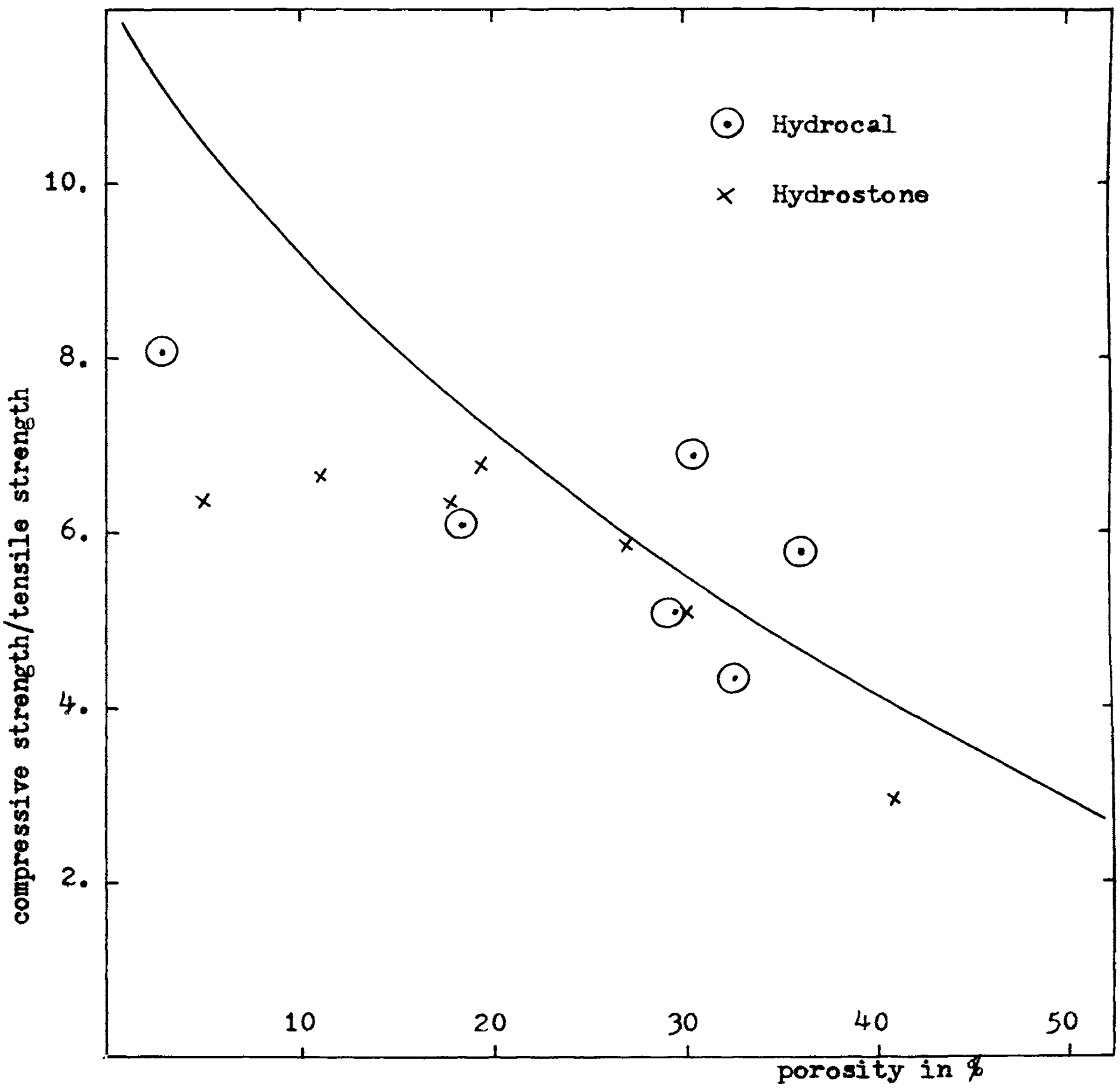


Figure 3-2. Compressive to tensile strength ratio of porous rock

compression. The data was obtained from reference 45. The zero porosity strength in this case was assumed to be 14,300 psi.

A series of tests were made with artificial porous rocks made of plaster. The descriptions and results of the tests are in Appendix D. The results are compared with theoretical curves in Figures 2, 5, 6, and 7.

All the comparisons made above indicate that the theory agrees with the experiments very well. By superposition of the solutions, the theory may be extended to three dimensional problems. For this matter, an extensive experimental study is required.

The case with solid inclusions shows some interesting results (see Appendix B). The tensile stress developed in matrix, when the composite is in compression, decreases as α increases. This indicates that, if the material breaks in tension rather than shear, the compressive strength increases with inclusion density, assuming that the inclusions are much stronger than the matrix. The tensile strength, on the other hand, decreases with increasing inclusion density. Thus "reinforcement" of material by adding stronger inclusions to it may apply only to compressive strength.

The increase in strength with inclusions has been shown in many experimental studies with metals (Reference 56, for example). No such studies have been made with brittle materials. Tests with artificial rocks, such as concrete, show that the strength decreases with increasing density of the inclusions (59). In order to compare such tests with the theory, not only the physical properties of individual constituents but also the porosity of matrix must be known (assuming no porosity for inclusions). For example, 10 per cent porosity of concrete is equivalent to about 50 per cent porosity of matrix when the inclusion-to-matrix volume ratio is about eight. The strength of the matrix should be reduced accordingly, in calculating the theoretical strength of the composite.

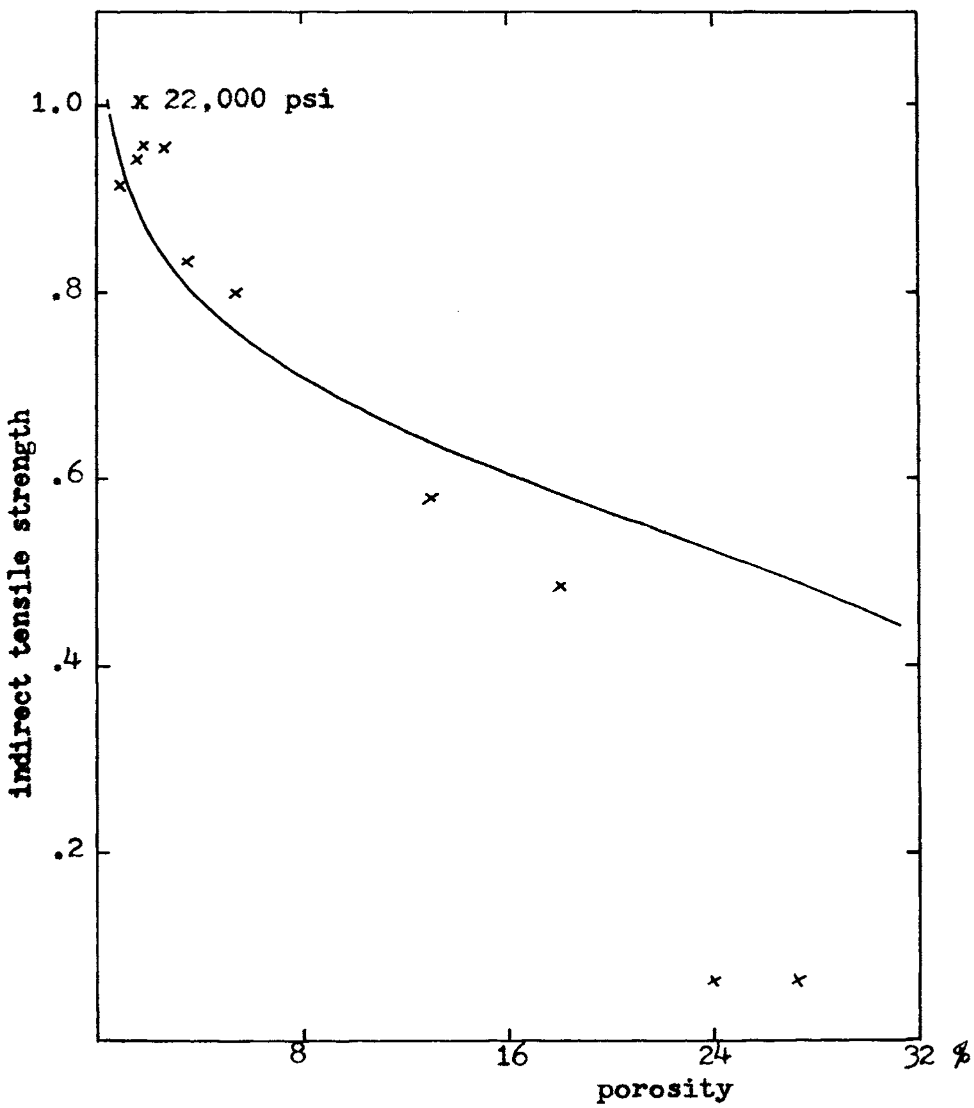


Figure 3-3. Indirect tensile strength of pressure-sintered N_iO vs. porosity

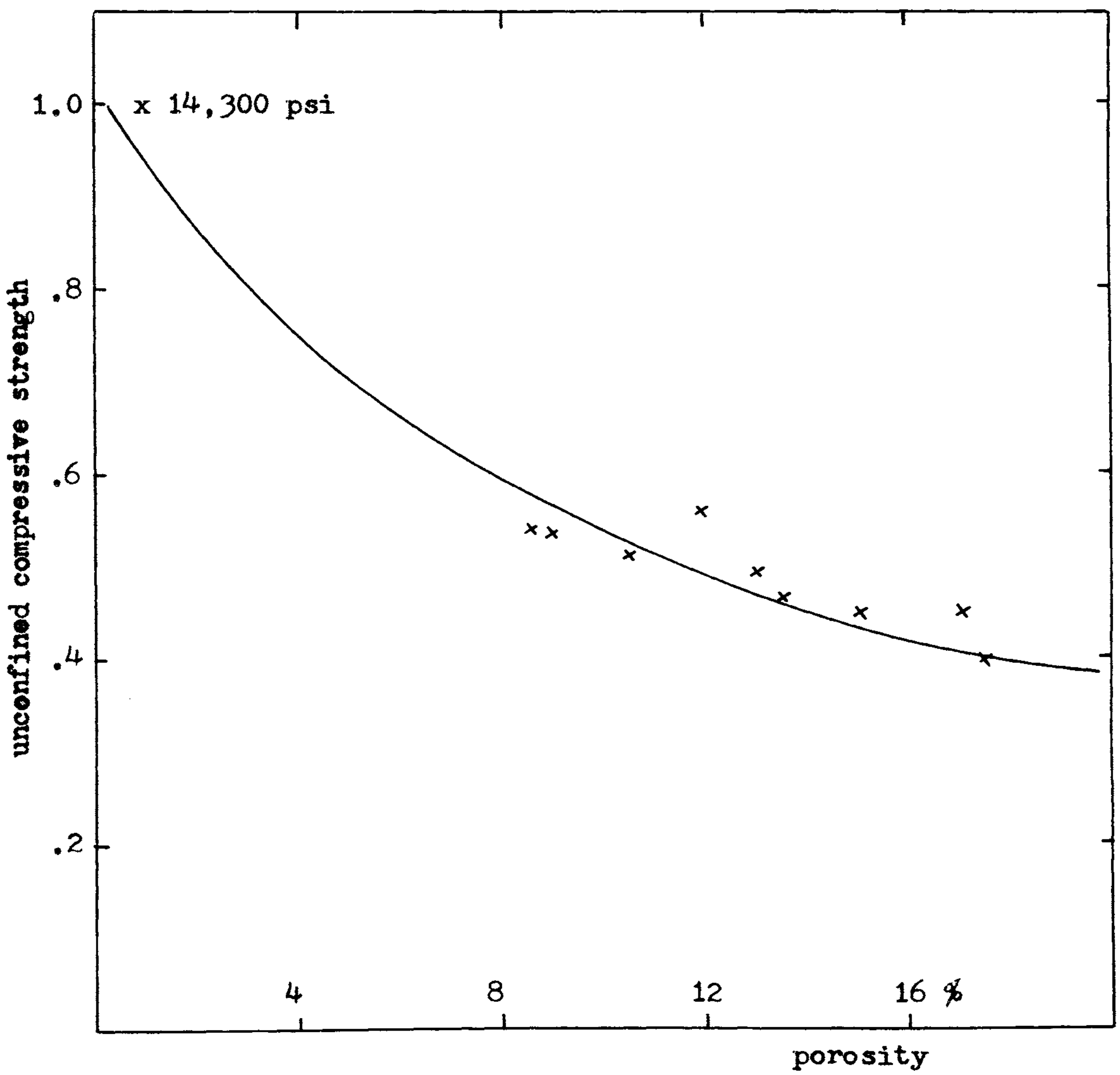


Figure 3-4. Unconfined compressive strength of Bunter sandstone vs. porosity

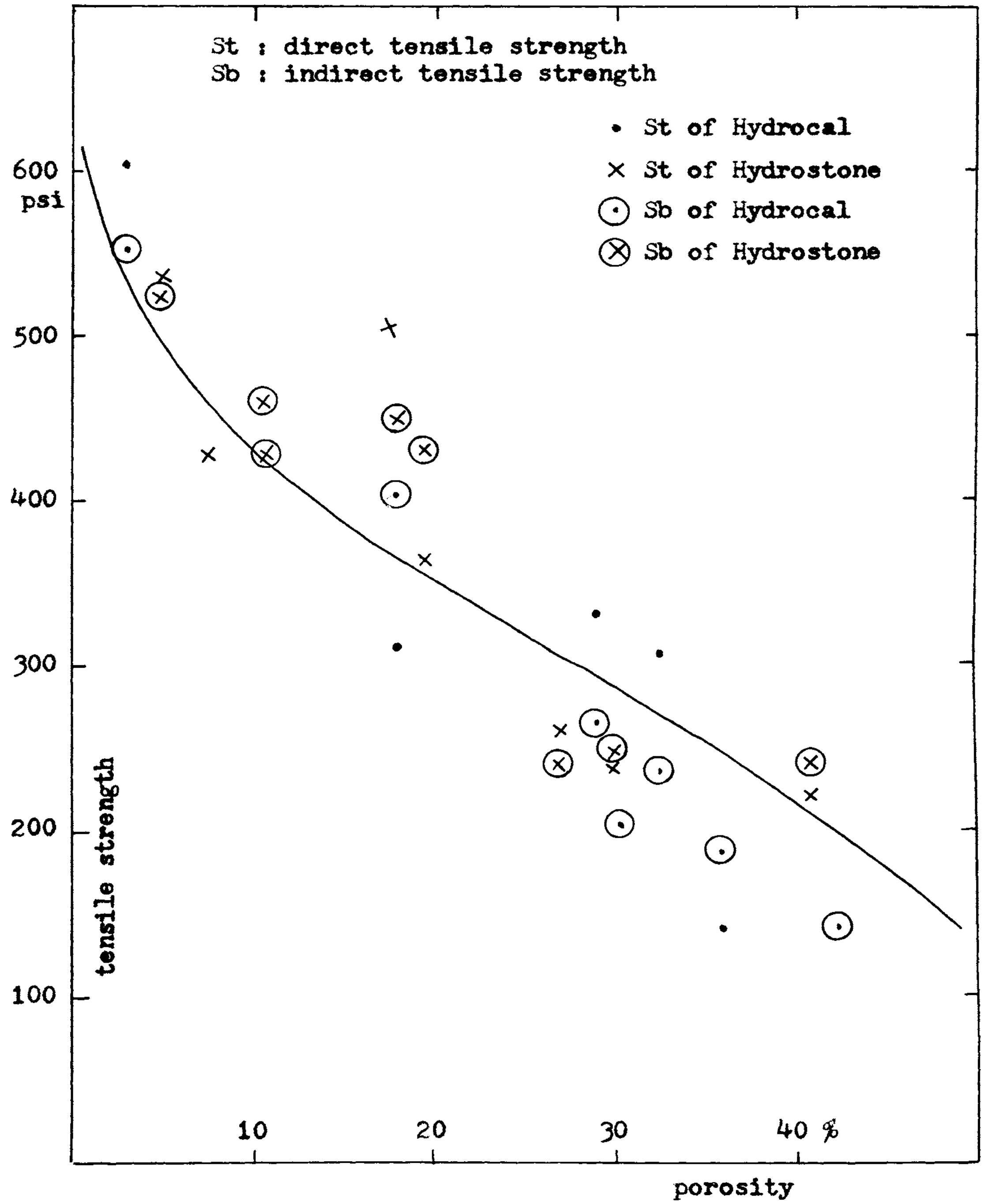


Figure 3-5. Direct and indirect tensile strengths of artificial rocks.

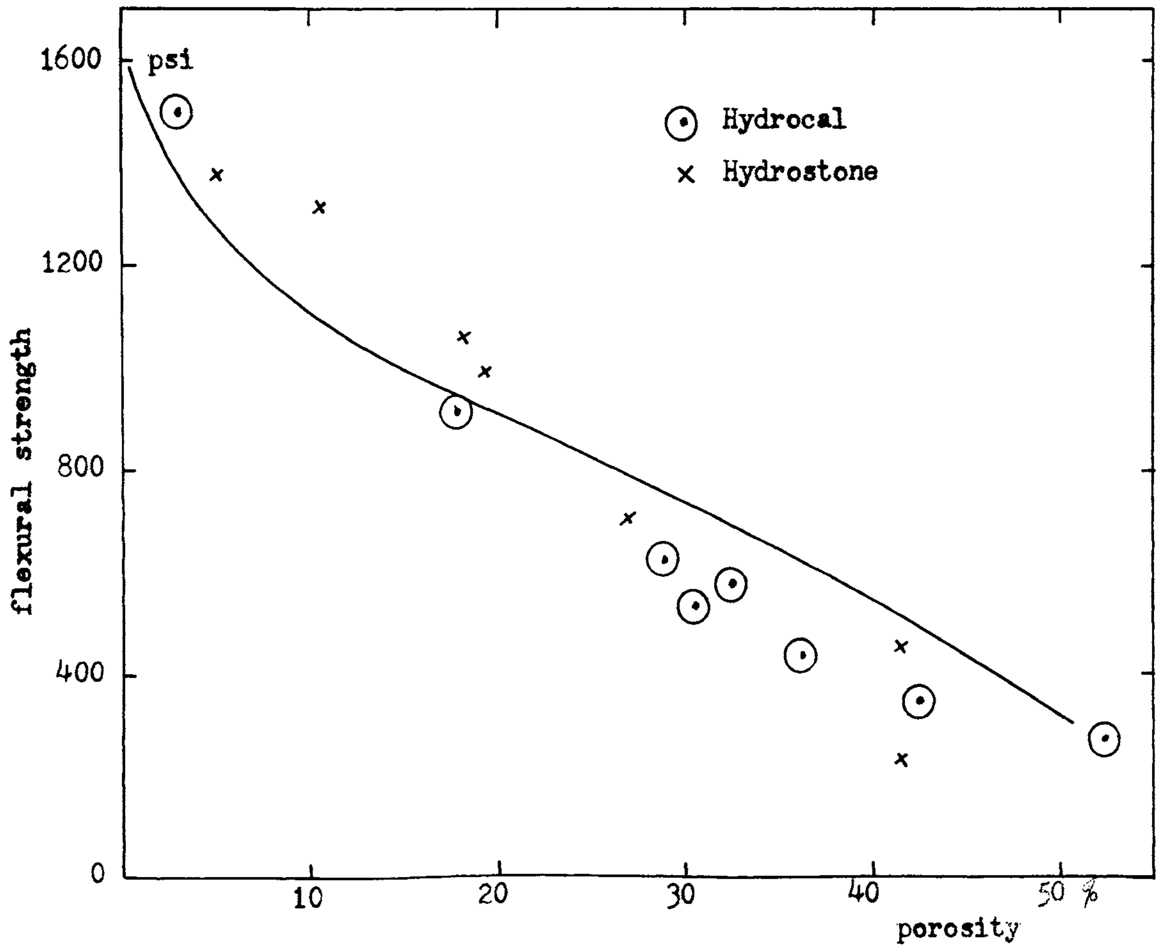


Figure 3-6. Variation of flexural strength with porosity

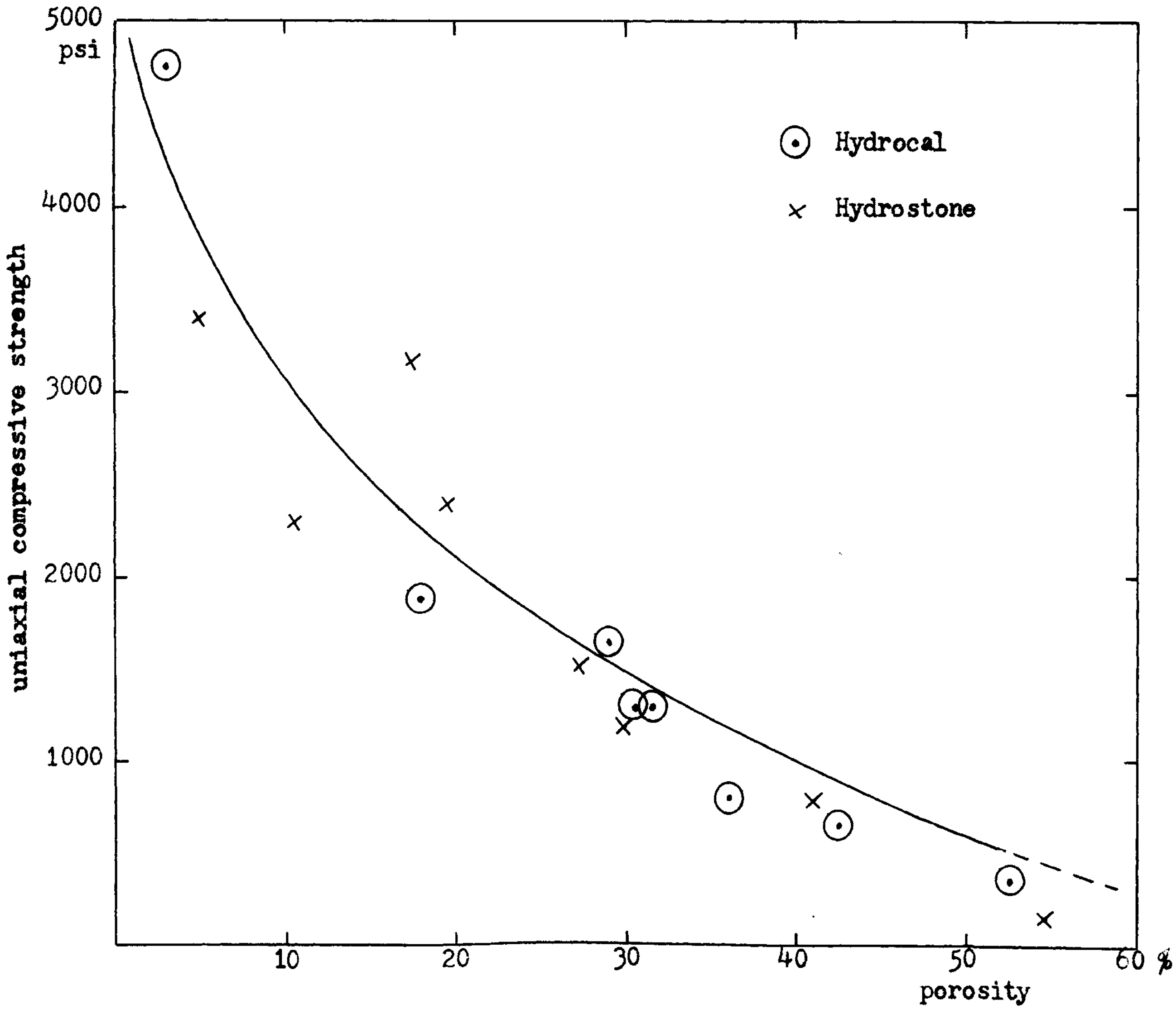


Figure 3-7. Variation of uniaxial compressive strength with porosity

CHAPTER IV

ELASTIC MODULI OF COMPOSITE

In the stress solutions obtained in Chapter II, we used elastic moduli of matrix E_1, of inclusion E_2, and of composite material E. We shall now look into the relations between these moduli in order to complete the investigation.

As mentioned in the literature review, very extensive investigations have been made on this subject. The most recent studies on this can be found in references 60 and 61. Simple and practical solutions have been proposed by a few authors. Two of these solutions together with the theoretical bounds for a two-phase composite are reviewed and a new approach to this problem is introduced. These are then compared with the data available in the literature and those obtained from a series of tests with artificial rocks.

A. Theoretical bounds. The lower bound is obtained by using the principle of least work in the theory of elasticity. This principle implies that if the tractions are completely prescribed over the surface of a body, the actual strain energy generated in the body is not greater than the strain energy calculated from the state of stress, i.e.,

$$\frac{1}{2E}\sigma_{ij}\sigma_{ij}\,V \le \frac{1}{2}\int_V \frac{\sigma_{ij}\sigma_{ij}}{E(\sigma_{ij})}\,dV$$

where $E(\sigma_{ij})$ is a function relating ϵ_{ij} to σ_{ij}. If we assume that the stress in the composite element is uniform and unidirectional,

$$\frac{1}{2}\int_V \frac{\sigma_{ij}\sigma_{ij}}{E(\sigma_{ij})}\,dV \le \frac{1}{2}\sigma_{ij}\sigma_{ij}\left[\int_{V_1}\frac{dV}{E_1} + \int_{V_2}\frac{dV}{E_2}\right]$$
$$= \frac{1}{2}\sigma_{ij}\sigma_{ij}\left(\frac{f}{E_1} + \frac{1-f}{E_2}\right)V$$

therefore,

$$E \ge E_L = \frac{1}{f/E_1+(1-f)/E_2} \tag{1}$$

where f is the volume ratio of matrix. Equation (1) is the same as the formula first used by Reuss (1929) in obtaining apparent moduli of heterogeneous aggregates (61).

The upper bound is found when the material lies in a uniform strain field. The theorem of minimum potential energy states that when the displacement components are completely specified on the surface of a body, the strain energy generated within the body due to the deformation does not exceed that calculated from the state of strain. Hence,

$$\frac{E}{2}\epsilon_{ij}\epsilon_{ij}V \le \frac{1}{2}\int_V \epsilon_{ij}\epsilon_{ij}E(\sigma_{ij})\,dV$$

If the strain is uniform throughout the material (12),

$$\frac{E_U}{2}\epsilon_{ij}\epsilon_{ij}V = \frac{1}{2}\epsilon_{ij}\epsilon_{ij}\int_V E\left(\frac{1-\nu-4\nu m+2m^2}{1-\nu-2\nu^2}\right)dV$$
$$= \frac{1}{2}\epsilon_{ij}\epsilon_{ij}V\left[\frac{(1-\nu_1-4\nu_1 m+2m^2)fE_1}{(1-\nu_1-2\nu_1^2)} + \frac{(1-\nu_2-4\nu_2 m+2m^2)}{(1-\nu_2-2\nu_2^2)}(1-f)E_2\right]$$

The value of m that minimizes the right hand side is found to be,

$$m = \frac{\nu_1(1+\nu_2)(1-2\nu_2)fE_1+\nu_2(1+\nu_1)(1-2\nu_1)(1-f)E_2}{(1+\nu_2)(1-2\nu_2)fE_1+(1+\nu_1)(1-2\nu_1)(1-f)E_2} \tag{2}$$

This gives the lowest upper bound of modulus E_U,

$$E_U = \frac{(1-\nu_1-4\nu_1 m+2m^2)fE_1}{(1+\nu_1)(1-2\nu_1)} + \frac{(1-\nu_2-4\nu_2 m+2m^2)}{(1+\nu_2)(1-2\nu_2)}(1-f)E_2 \tag{3}$$

When $\nu_1=\nu_2$, E_U becomes

$$E_U = fE_1 + (1-f)E_2 \tag{4}$$

which is the formula derived by Voigt in 1910 (61).

The elastic moduli of a composite with cylindrical inclusions with uniform cross section is equal to E_u when the load is applied in the direction normal to the cross section and is equal to E_L when the load is applied parallel to the cross section. Kumazawa (61) observed, however, that under high-pressure static compression, the behavior of rock is better simulated by E_L rather than E_u.

B. Paul's approximation formula. Paul (1960) used a unit cube containing one inclusion as a representative element (Figure 1). Assuming that the cross section originally normal to the axis of applied force remain normal, and that the strain ϵ_i is uniform over such a cross section, the total force on the section can be expressed as

$$F = E_1 \epsilon_i A_1 + E_2 \epsilon_i A_2 = \epsilon_i (E_1 + (E_2 - E_1) A_2)$$

The total deformation δ of the cube is

$$\delta = \int_0^1 \epsilon_i (x) dx = F \int_0^1 \frac{dx}{E_1 + (E_2 - E_1) A_2}$$

but since $E = F/\delta$,

$$\frac{1}{E} = \int_0^1 \frac{dx}{E_1 + (E_2 - E_1) A_2(x)} \qquad (5)$$

where the function $A_2(x)$ is dependent on the shape of the inclusion.

For a cubic shape inclusion, equation (5) yields;

$$\frac{E}{E_1} = \frac{E_1 + (E_2 - E_1)(1-f)^{2/3}}{E_1 + (E_2 - E_1)(1-f)^{2/3}\{1-(1-f)^{1/3}\}} \qquad (6)$$

The assumption of sectionally uniform strain is actually closer to the assumption of non-uniform strain than that of uniform strain in z-direction. The assumption used in this formula is, in reality, a

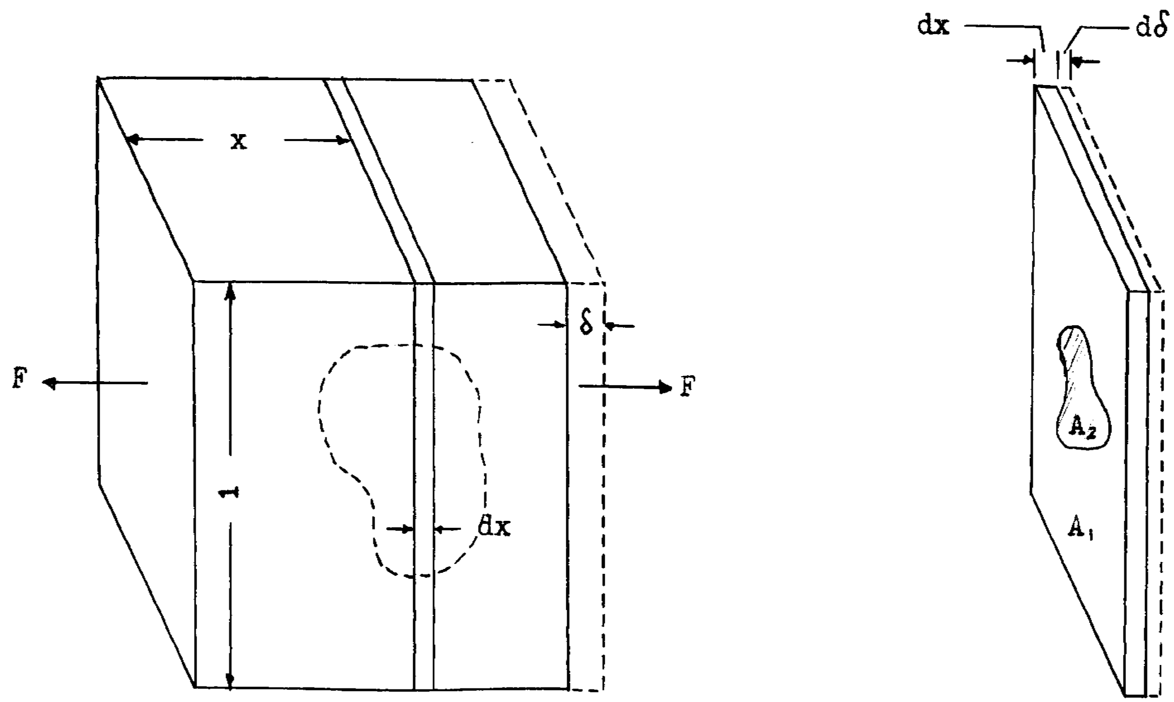


Figure 4-1. Paul's unit cube

rough approximation of uniform stress in the media. Thus, in most cases, the formula is merely an improvement of the upper bound.

C. Greszczuk's expression. Assuming the unit element to be as shown in Figure 2, and the strain to be uniform in z-direction, the effective or apparent Poisson's ratio becomes,

$$\nu = \nu_1 f + \nu_2 (1-f) \tag{7}$$

The volume change of an elastic solid in hydrostatic stress field is

$$\Delta V = \frac{1-2\nu}{E} \sigma_{ii} \tag{8}$$

but

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{9}$$

where

$$\Delta V_1 \fallingdotseq \frac{(1-2\nu_1)}{E} f \sigma_{ii} \tag{10}$$

$$\Delta V_2 \fallingdotseq \frac{(1-2\nu_2)}{E} (1-f) \sigma_{ii}$$

Combining equations (7) through (10) and solving for E, we obtain:

$$\frac{E}{E_1} = \frac{E_2\{1-2\nu_2(1-f)-2\nu_1 f\}}{f(1-2\nu_1)E_2+(1-f)(1-2\nu_2)} \tag{11}$$

The effective shear modulus is determined by using effective E and ν, i.e.,

$$\mu = \frac{E}{2(1+\nu)} \tag{11a}$$

The unit element indicates the model to be of an anisotropic nature, the inclusion density in z-direction being greater than that in lateral direction. Hence this equation gives somewhat higher values of E than actual E when the concentration of the inclusion is small, and it gives the lower values when concentration is high.

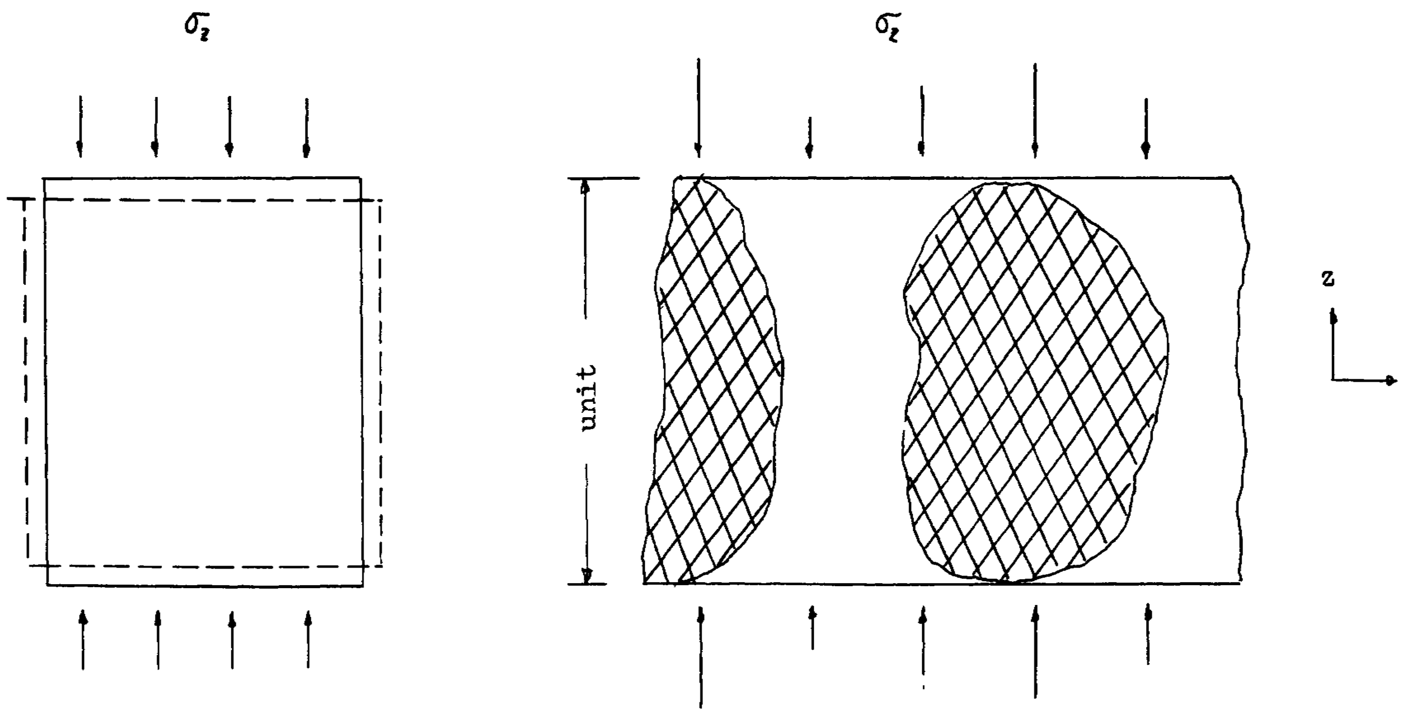


Figure 4-2. Greszczuk's unit element

It should be also noted that because the formula has been derived from $\nu - \Delta V$ relation, the effective Young's modulus is highly sensitive to the ratio of ν_2/ν_1. As will be shown later, this formula gives good fit th experimental data only when ν_2/ν_1 is very small or very large. Since the Poisson's ratio of most rock constituent minerals are fairly constant within the range of .2 to .3 (62), this formula may not be the best to use for rocks.

D. A new approach. Since the theoretical work on the subject of this chapter has been carried out to almost beyond any more improvement, our prime objective remains to find a formula for effective elastic moduli of composite material. The derivation of this formula should be based on theoretically reasonable assumptions, and such formula should, a) be simple and practical to use, and b) fit experimental data better for a wider range of property variation than other formulas.

In order to simulate a quasi-isotropic condition, we shall use a unit element similar to Paul's (Figure 3). As in the case of stress problem, we will assume that the boundaries of elements will undergo uniform displacement. This implies neither uniform stress nor uniform strain. However, it enables us to use sectionally uniform strain in the direction of load for a uni-axial load field. We devide the element into columns with a very small cross-sectional area. The strain in z-direction within a column can be assumed to be uniform, except those containing parts of the inclusion. If we replace these heterogeneous columns with homogeneous ones that have equivalent elastic properties, then we will be able to use the theorem of minimum potential energy for the system.

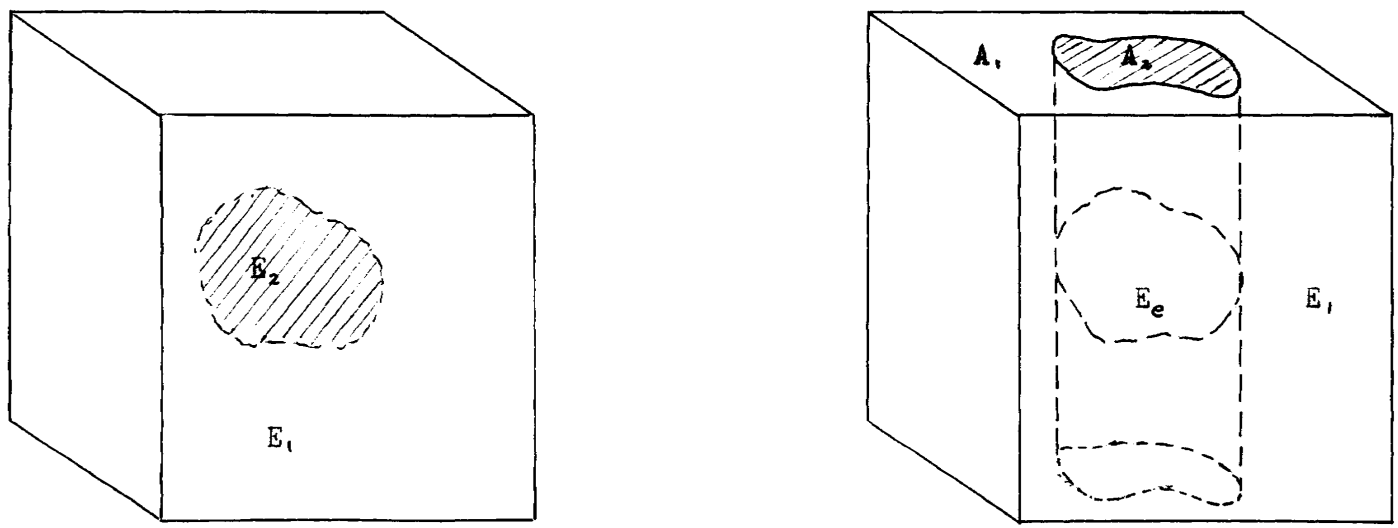


Figure 4-3. Replacement of inclusion

In order to find such equivalent elastic modulus, we assume that the stress is constant throughout individual heterogeneous columns. From the principle of least work,

$$E_{ei} = \frac{E_1 E_2}{E_1 z_i + E_2(d - z_i)}$$

where E_{ei} is equivalent modulus of individual column. The equivalent modulus of the column containing the entire inclusion is obtained by averaging over the area,

$$\begin{aligned} E_e &= \frac{E_{ei} A_i}{A_i} \\ &= \frac{1}{A}\int_A \frac{E_1 E_2}{E_2 d + (E_1 - E_2) z}\, dz \\ &= \frac{E_1}{a}\int_A \frac{\beta}{\beta d - (1-\beta) z}\, dz \end{aligned} \tag{12}$$

The effective modulus of the element becomes then,

$$E = (E_1 A_1 + E_e A_2)/(A_1 + A_2) \tag{13}$$

Equation (13) has been obtained through utilizing both theorems used in deriving the bounds, and can be used for any shape of inclusion. However, we can readily perceive that this formula will fit best if the inclusion is granular, i.e., if the three dimensions of the inclusion are about the same.

For a composite with spherical inclusions, equation (12) becomes,

$$\frac{E_e}{E_1} = \frac{2\beta}{\alpha^2(\beta-1)^2} \log \frac{\beta}{\beta-(1-\beta)\alpha} + \frac{2\beta}{\alpha(1-\beta)} \tag{14}$$

and, $$\frac{E}{E_1} = \left(\frac{E_e}{E_1} - 1\right)\frac{\pi}{4}\alpha^2 + 1 \tag{15}$$

For cubic inclusions,

$$\frac{E_e}{E_1} = \frac{\beta}{\beta + (1-\beta)\alpha}$$

For cavities, E_2 is taken to be zero, hence E_e=0. This is reasonable since the deformation of rock containing a spherical void is the same as that containing a crack normal to the load (63). Equation (13) becomes, for void inclusions (spherical),

$$E = E_1 \left(1 - \frac{6}{\pi\alpha}\eta\right) \tag{16}$$

where η denotes porosity.

The effective Poisson's ratio is obtained simply by combining E, E_1, and E_2. From equations (8), (9) and (10),

$$\frac{1-2\nu}{E} = \frac{1-2\nu_1}{E_1}f + \frac{1-2\nu_2}{E_2}(1-f)$$

Hence,

$$\nu = \frac{1}{2}\left\{1 - \frac{(1-2\nu_1)f\beta \quad (1-2\nu_2)(1-f)}{E_2}E\right\} \tag{17}$$

When $E = E_1 = E_2$

$$\nu = f\nu_1 + (1-f)\nu_2 \tag{18}$$

The effective shear modulus can be found by using equation (11a).

E. Comparison with available experimental data. The values calculated from equation (15), assuming spherical inclusions, were compared to the experimental results in order to examine the accuracy of the expression. Figure 4 shows a comparison of predicted E to experimental data obtained by Nishimatzu and Gurland (56) for an alloy system of tungsten carbide (inclusion) and cobalt (matrix). The following values are taken from reference 56:

$E_1 = 30 \times 10^6$ psi

$E_2 = 102 \times 10^6$ psi

$\nu_1 = 0.3$

$\nu_2 = 0.22$

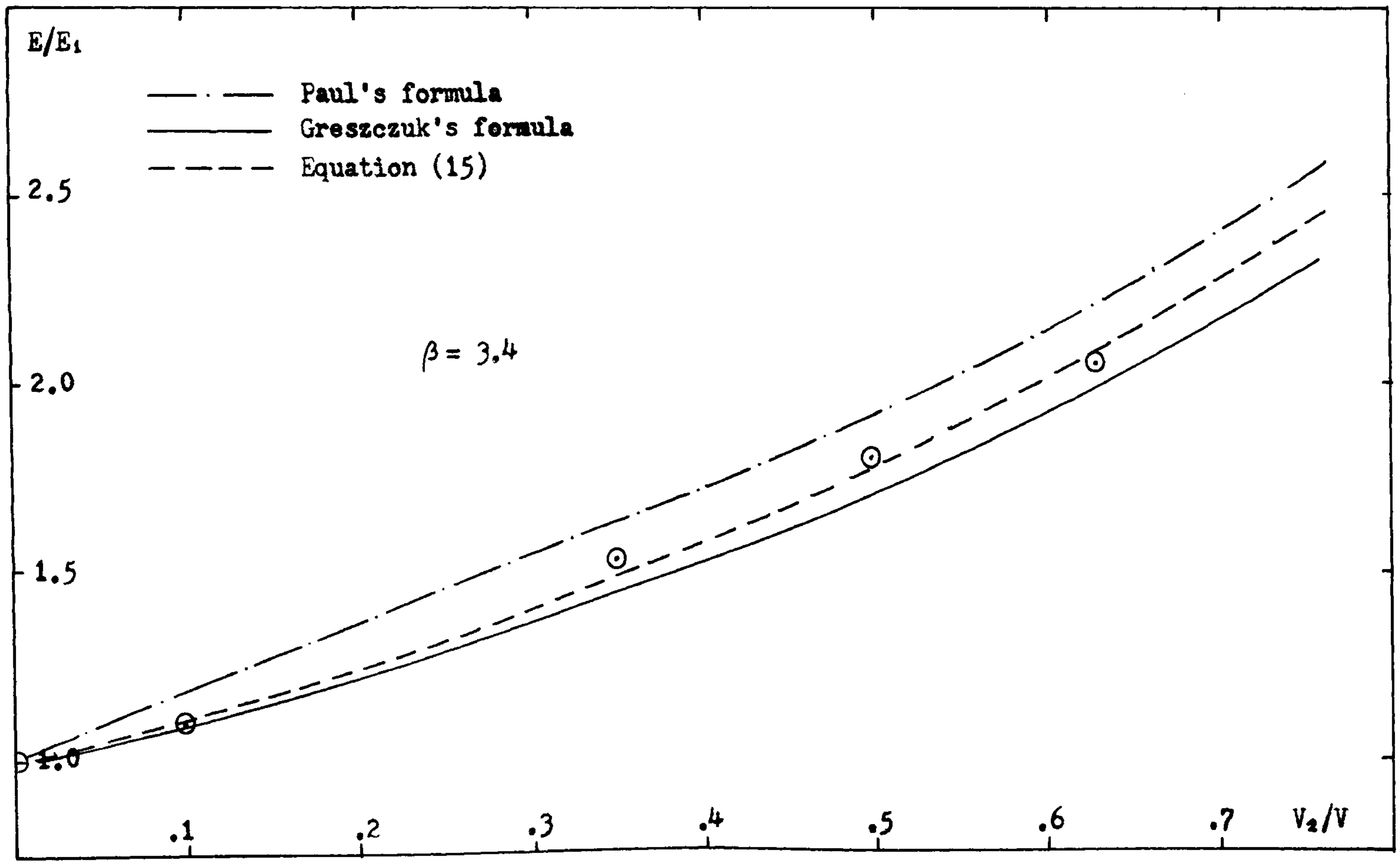


Figure 4-4. Young's modulus of cobalt-tungsten carbide composites

As is readily seen, the formula correlates the experimental data quite well.

In Figures 5 to 9, the formulas are compared to the data by Mandel and Dantu (40). The material used for this experiment was an epoxy type resin called Araldite with five different materials for inclusions. The average values for matrix are $E_1=.45\text{x}10^6$ psi and $\nu_1=.40$. The elastic properties of the inclusions are (40),

Inclusions	E_2	ν_2
Steel balls	31.2×10^6 psi	0.30
Diorite	14.6×10^6 psi	0.20
Limestone	11.2×10^6 psi	0.25
Sandstone	8.3×10^6 psi	0.25
Lead balls	3.3×10^6 psi	0.40

In general, the equation (15) seems to agree to the experimental data better than the other two. The most remarkable example is seen in Figure 9.

A further verification of the approximation formula is shown in Figure 10. The data was obtained from reference 65, where the experiments were made on a tungsten alloy with copper. The elastic moduli of tungsten and copper were 59.0×10^6 psi and 17.56×10^6 psi, respectively.

The main concern of this paper is the behavior of granular rocks. Artificial rocks made of plaster and water were used in the experimental verification of the theory since their properties can be more easily controlled than with real rocks. The types of plaster used and test results are shown in Appendix C.

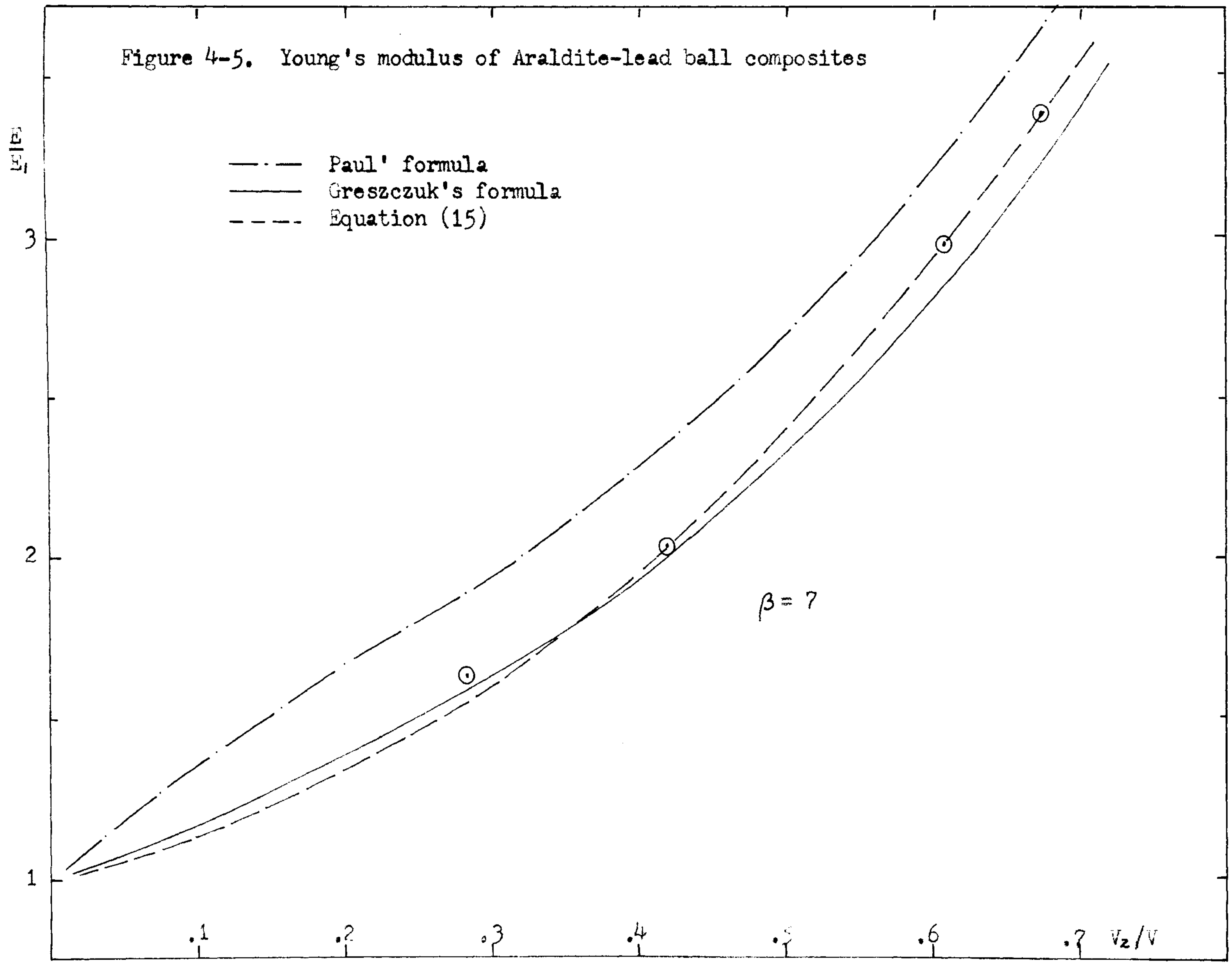


Figure 4-5. Young's modulus of Araldite-lead ball composites

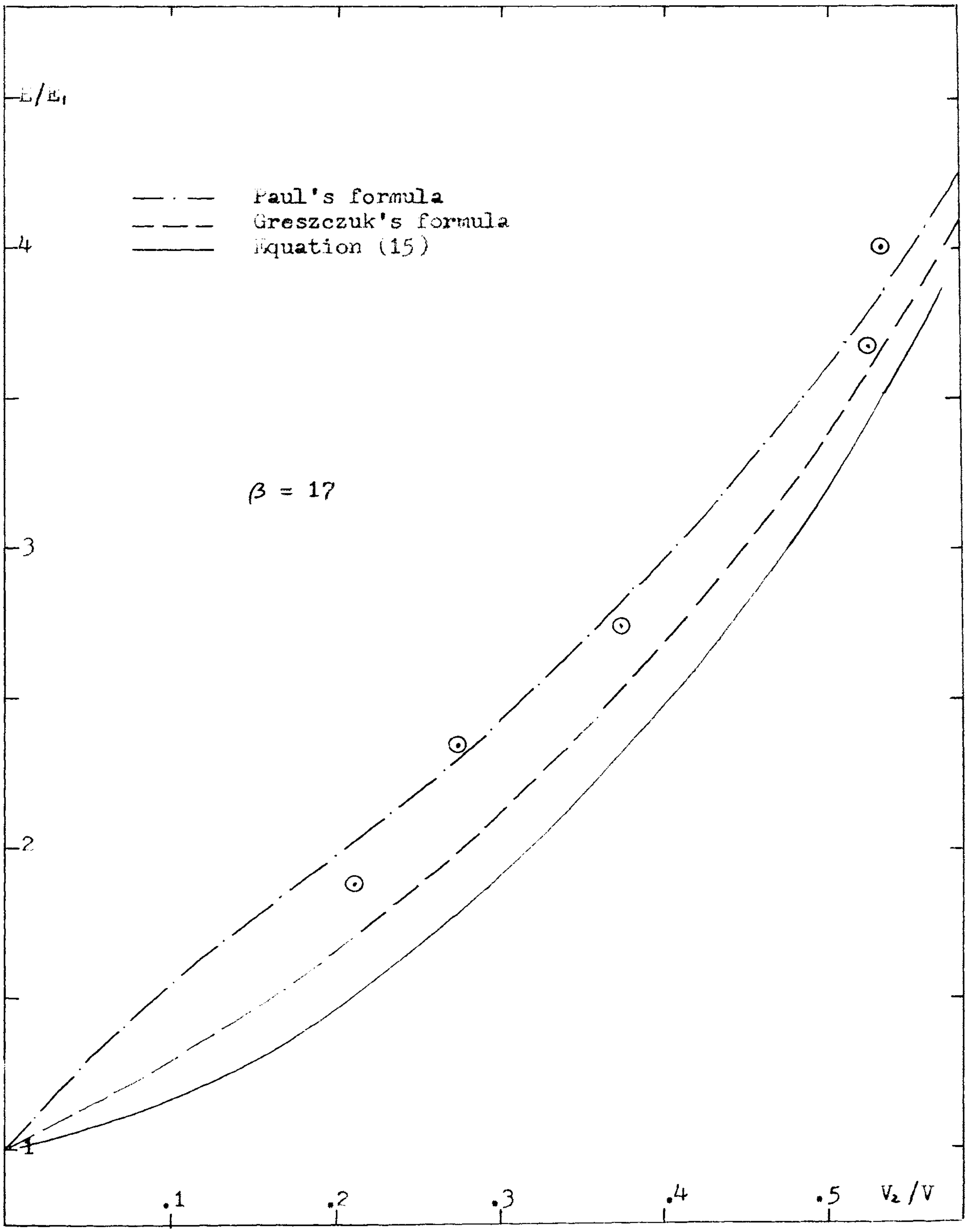


Figure 4-6. Young's modulus of Araldite-sandstone composites

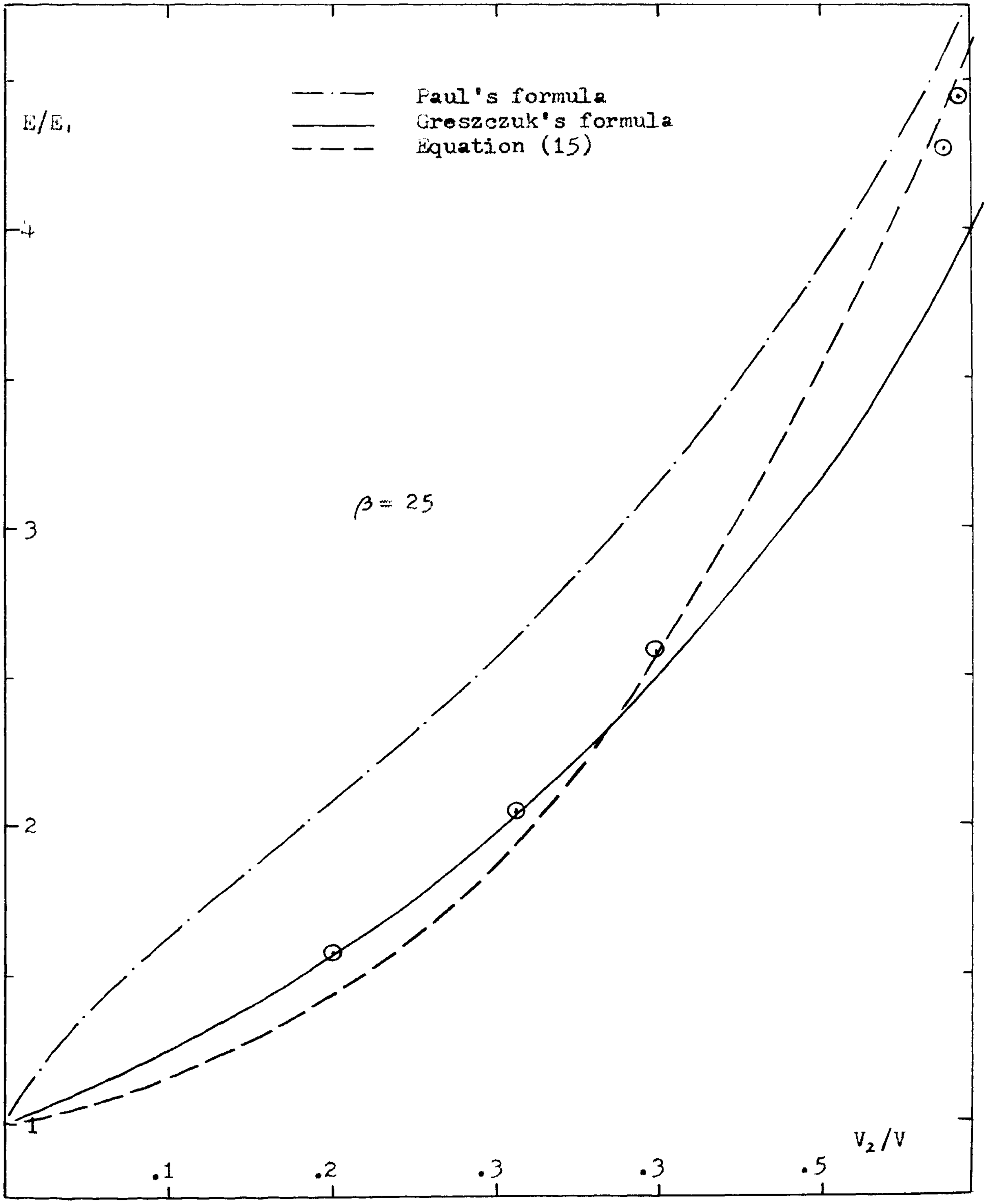


Figure 4-7. Young's modulus of Araldite- limestone composites

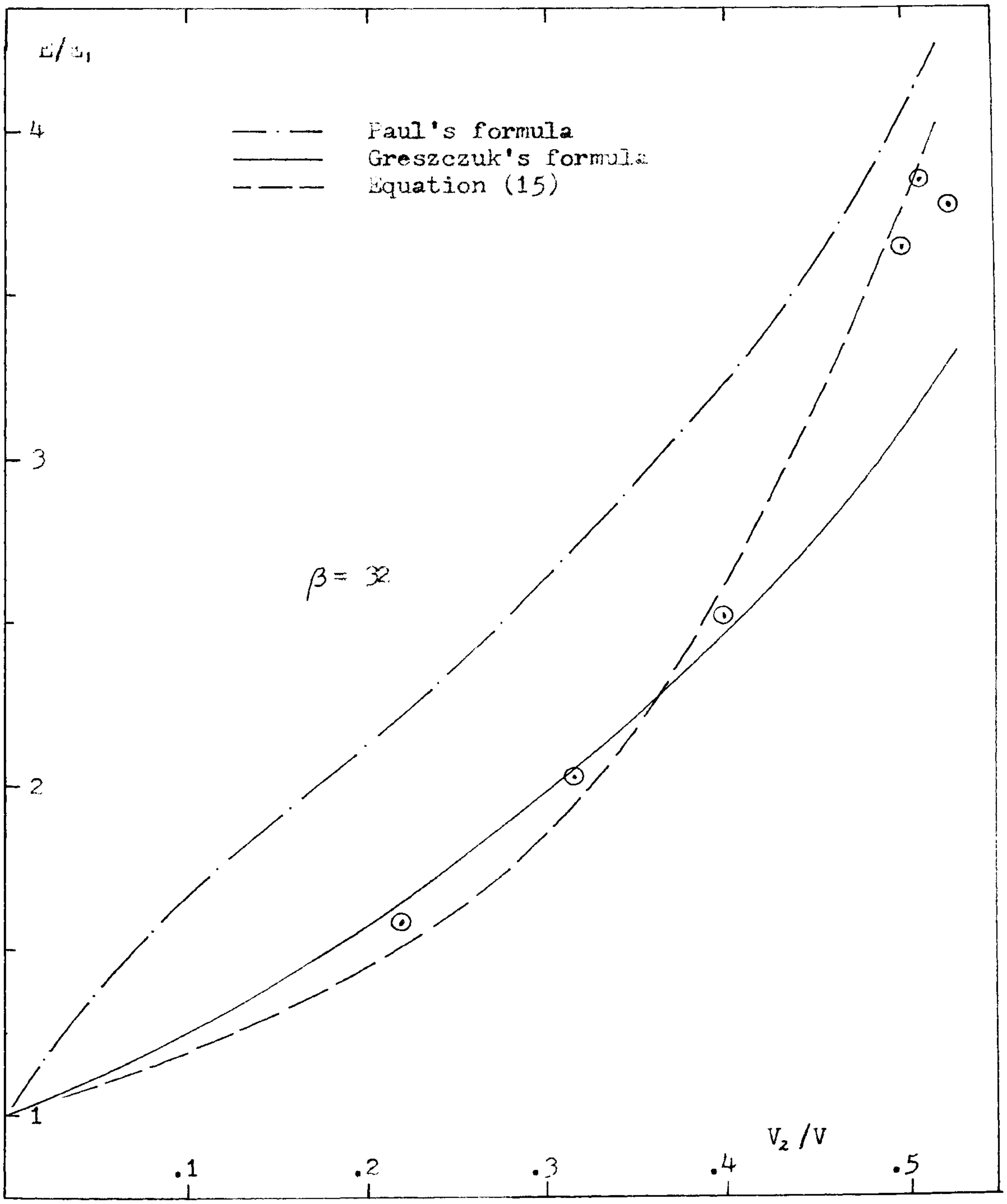


Figure 4-8. Young's modulus of Araldite-diorite composites

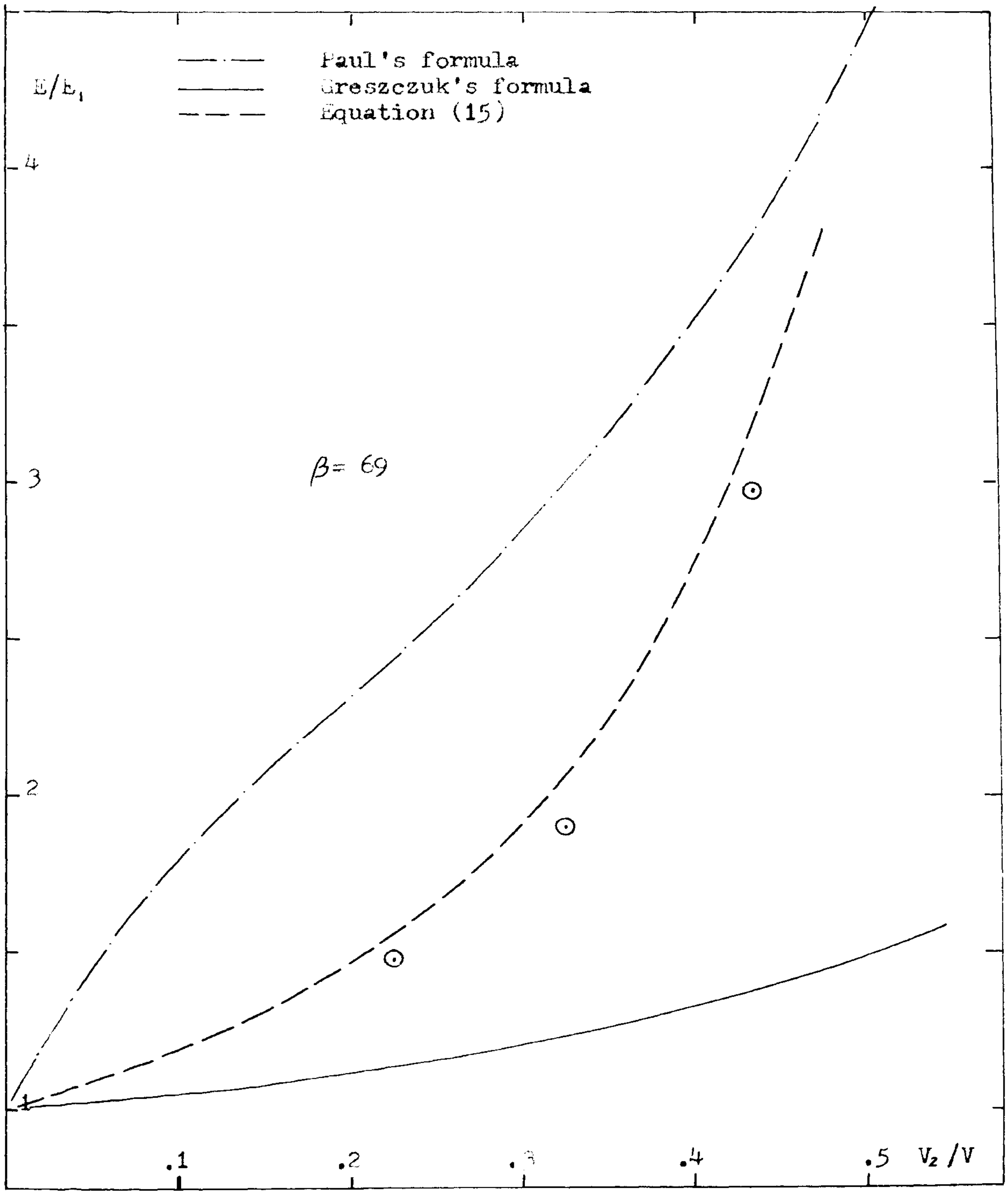


Figure 4-9. Young's modulus of Araldite-steel ball composites

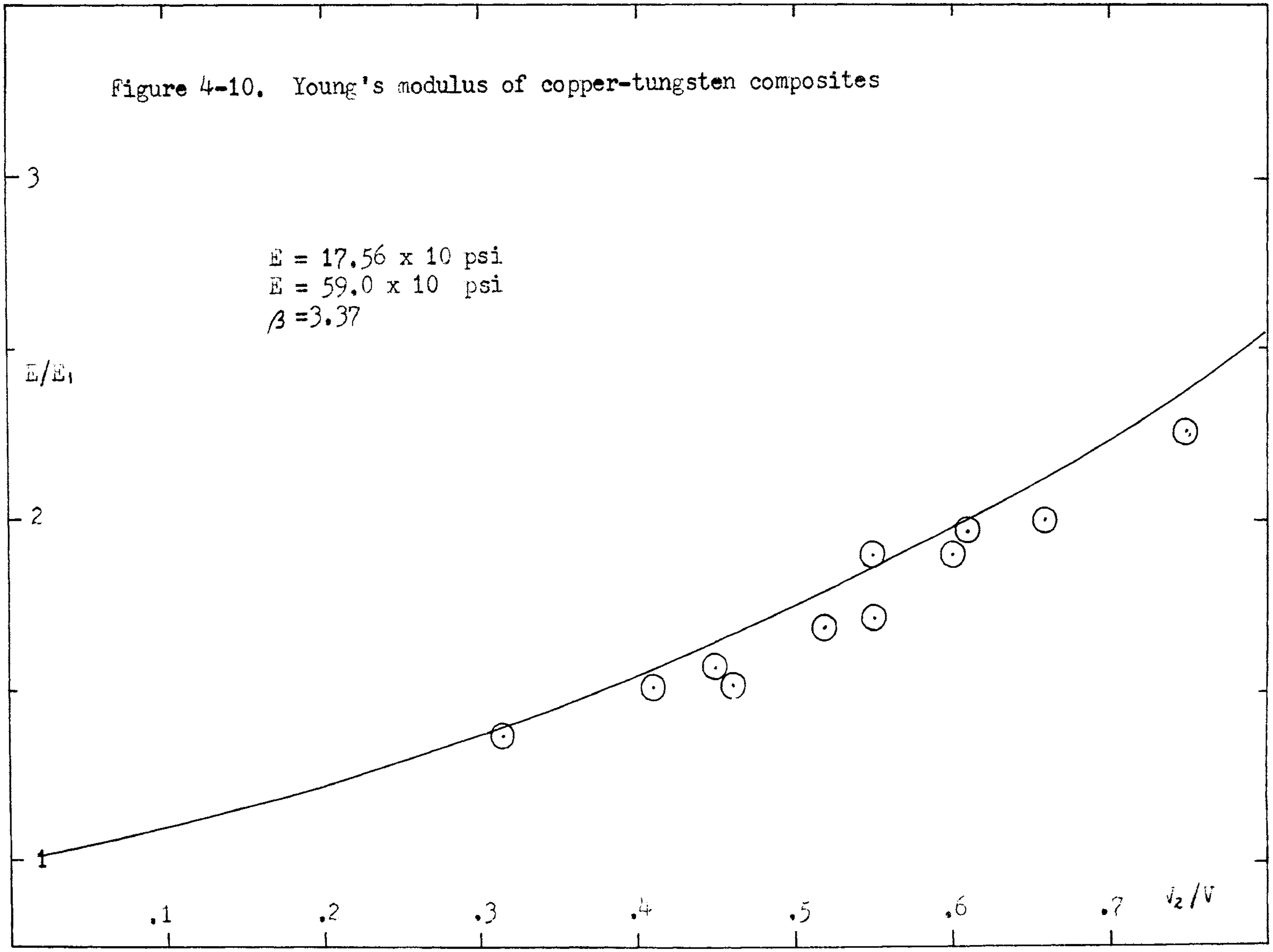


Figure 4-10. Young's modulus of copper-tungsten composites

The ratio of measured Young's modulus to E_1, assuming E_1 to be 2.5×10^6 psi, is plotted against porosity in Figure 11. The predicted values by equation (16) agree remarkably well with the data in view of the assumptions made above. It was observed that when the porosity exceeds 40 per cent, the plaster specimens behaved quite non-elastically. This may be the probable cause of lower values of experimental data than those predicted in high porosity ranges.

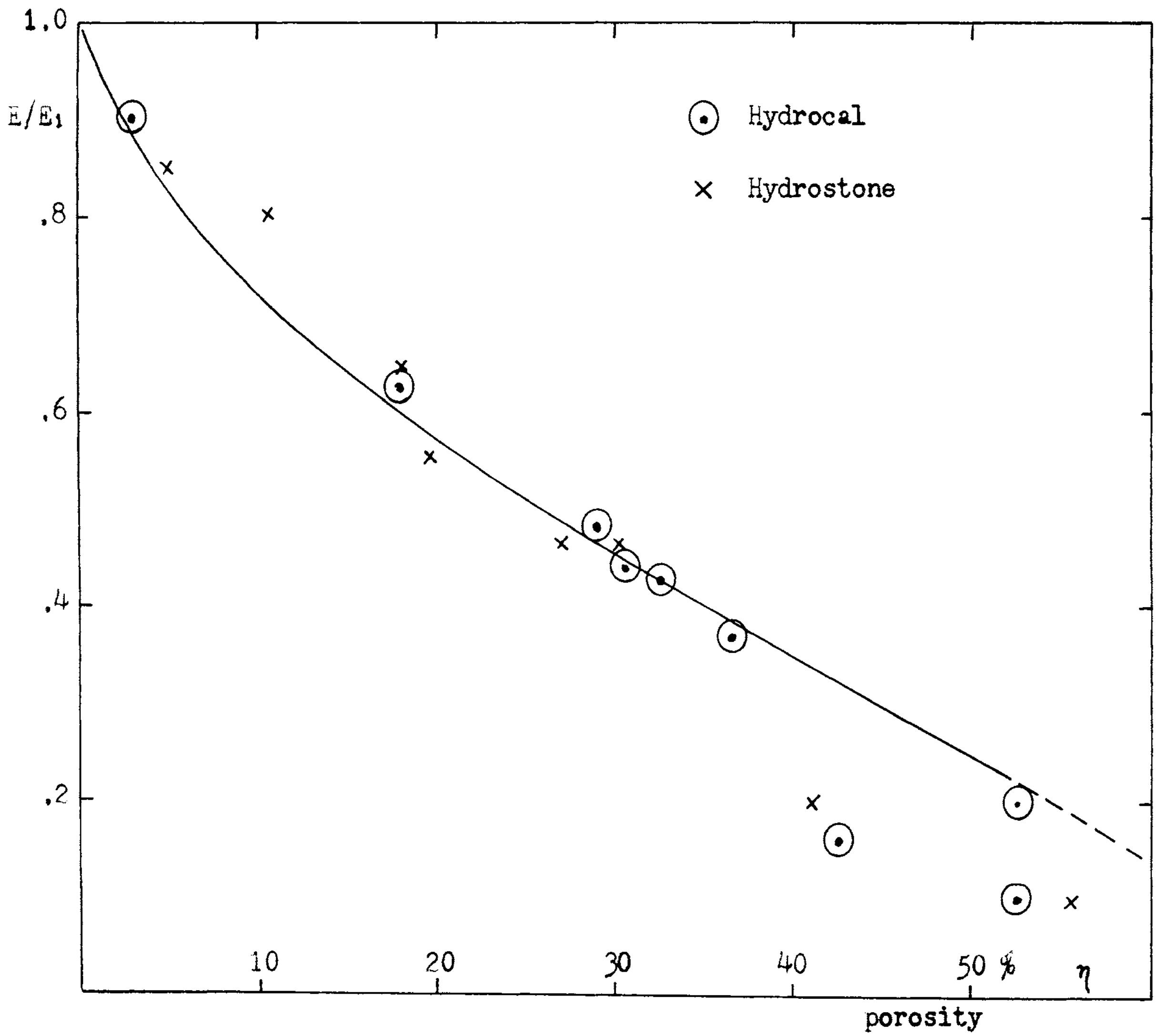


Figure 4-11. Young's moduli of porous materials

CHAPTER V

SUMMARY, CONCLUSIONS, AND SUGGESTIONS FOR FUTURE INVESTIGATIONS

Analytical solutions for stresses in elastic composites with spherical inclusions have been obtained on the basis of mathematical theory of elasticity. The basic assumptions used were that individual material composing the composites is homogeneous, isotropic, and perfectly elastic, and that the boundaries of representative elements deform uniformly.

The solutions indicate that the failure criteria of composite elastic materials are complicated functions of ν_1, E_1, S_1, α, β, γ, and δ. Some significant results are as follows:

a) when the composite is extended (compressed), the maximum tensile (compressive) stress on the grain boundary increases as α, β, and γ increase;

b) when compressed, the maximum tensile stress and maximum extension decrease very slowly with increasing α and γ, but increase somewhat as β increases;

c) for void inclusions, the tensile stress developed due to compressive load increases as α increases;

d) the maximum shear stress varies similarly to the maximum principal stress.

Combining these solutions with Griffith's microcrack theory provides a new failure criteria for brittle granular rocks. With this theory, most phenomena in failure of brittle rock that were not possible with other theories can be easily explained without assumptions, such as friction on crack surfaces or hardening of cracks.

For example:

a) The ratio of compressive to tensile strength is not exactly eight, but is a function of the properties and their ratios of individual minerals composing the rock. It may vary from less than eight to greater than 100.

b) High stress concentrations along the grain boundaries are responsible for frequent failures along the grain boundaries. This eliminates the assumption that micro cracks concentrate along such boundaries to explain the failure.

c) The inter-granular or intra-granular failures are determined by the physical properties of the composite.

d) The stress rises very rapidly as γ increases. This may explain why rocks are weaker when wet.

The theory also agrees with experimental data quite well. Thus the theory appears to provide a near perfect model for brittle and elastic granular rocks, from both the mathematical and experimental viewpoints.

As part of this thesis, approximation formula for effective elastic moduli was obtained through combination of two theorems from the theory of elasticity. From the comparisons with test results, the formula was proven to give better approximation than other formulas.

A more extensive experimental study with granular rocks is suggested for further verifications of the validity of the theories obtained in this paper. Extension of the theory into three dimensional failure criteria requires a verification. This may be done by comparing tri-axial test results with the solutions superposed in three directions. The superposition may be made easier by direct use of the digital computer. Rock usually becomes plastic at a very high confining pressure. Hence there should be certain limits to the applicability of the theory. The three dimensional criteria may be used to find such limits for rocks.

By applying different boundary conditions, stresses in an anisotropic material can be analysed. This may be done by using a parallelopiped instead of cubic element. In this case, however, the stress functions might be different from those used here, depending on the boundary displacement function.

Some rocks contain cracks (macro size) along the grain boundaries due to pre-existing stresses. Analysis of such composites may be made by assuming imperfect bonding or no bonding at all between the grain and matrix. The resulting solutions might give a better description of real granular rocks.

APPENDICES

APPENDIX A

COMPARISON WITH SOLUTIONS BY FINITE ELEMENT TECHNIQUE

The theoretical solutions are compared to the solutions obtained by means of finite element technique. The composite cube would have been an ideal model to compare displacements of spherical boundaries to those obtained by theoretical solutions, but because the computer program for three dimensional analysis was not available, a cylinder containing a spherical inclusion was used as a model. One disadvantage of the finite element method is that it does not give stresses at the boundary. Thus it requires finer meshes along the boundaries to obtain better approximations. Due to limited computer time allowed, however, a very simple mesh (Figure 1) was used.

All solutions are for $\alpha = 5/8$. The displacements of outer and grain boundaries are also compared. For the purpose of comparison, the stresses obtained from theoretical solutions were converted to those in the cylindrical coordinate system, and the displacements from finite element analysis were converted to those in the spherical coordinate system. All displacements and stresses are calculated at the points on a vertical plane containing the center of the inclusion.

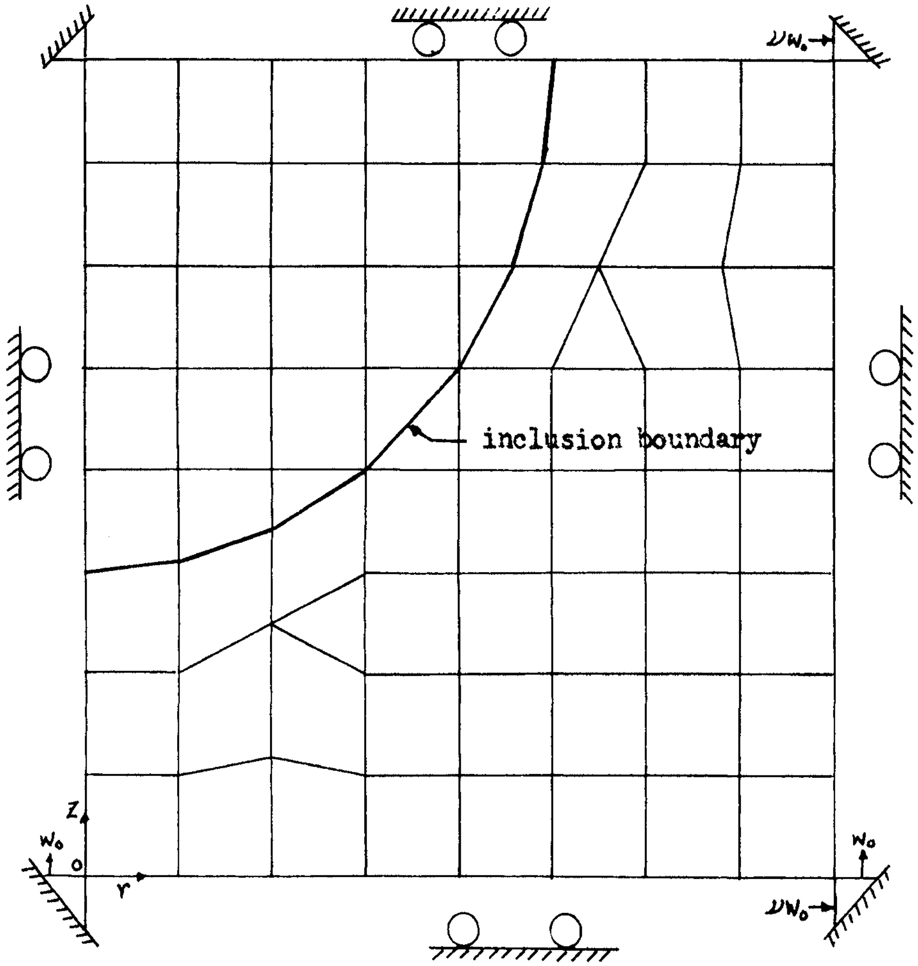


Figure A-1. Mesh used for finite element analysis

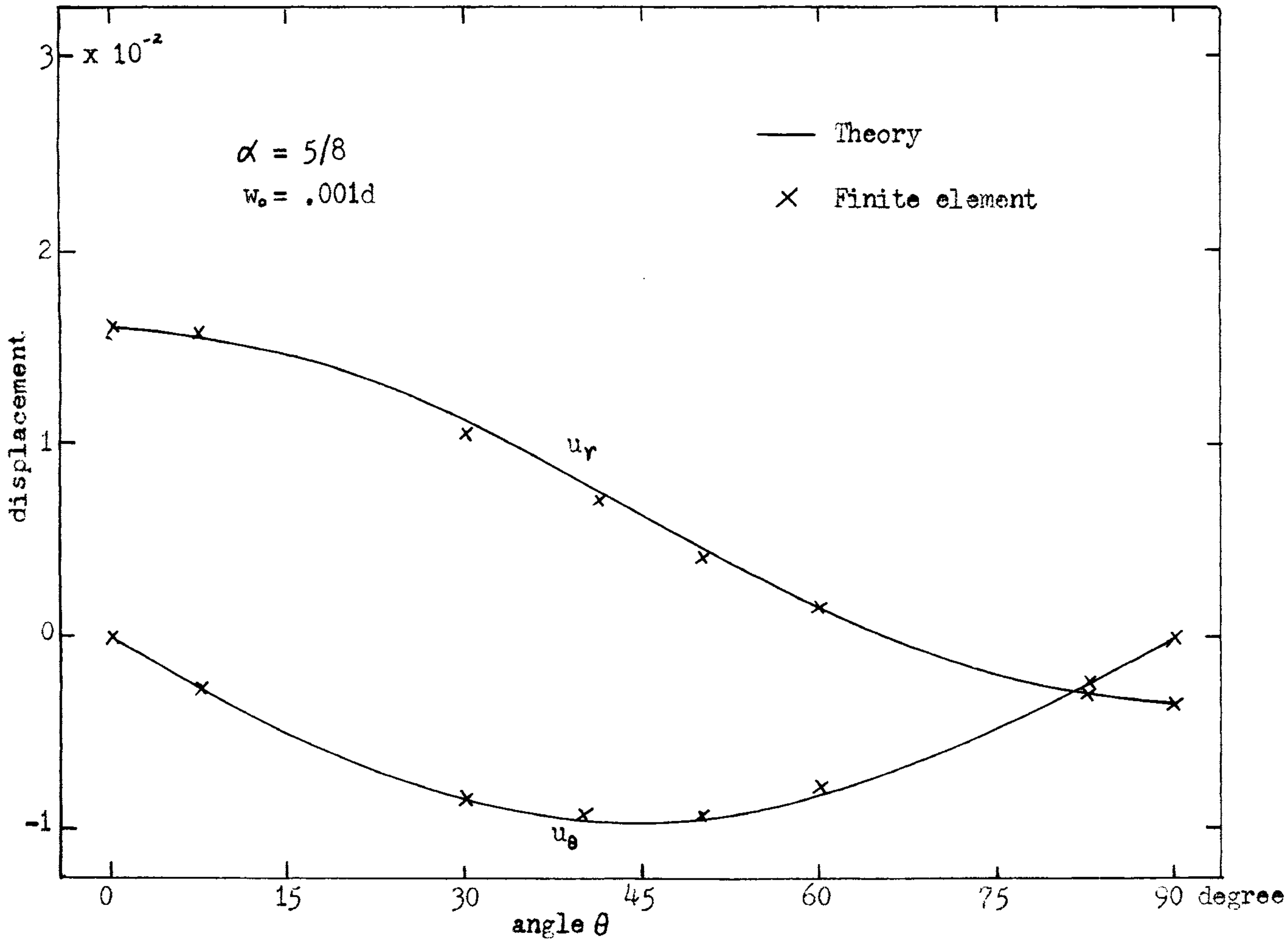


Figure A-2. Displacements at r = d for β= 10 and γ=1

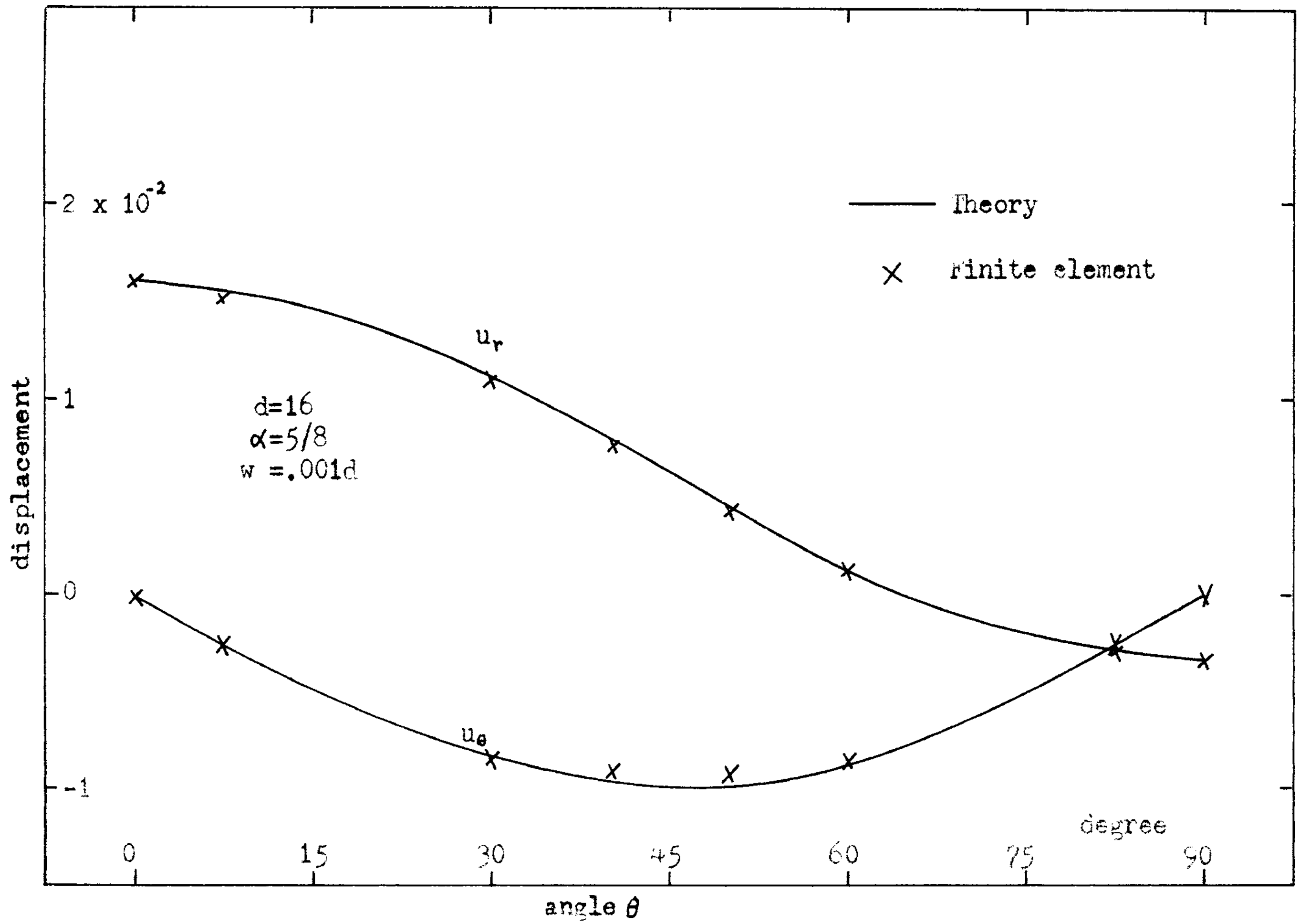


Figure A-3. Displacements at r = d for β=2. and γ=1.

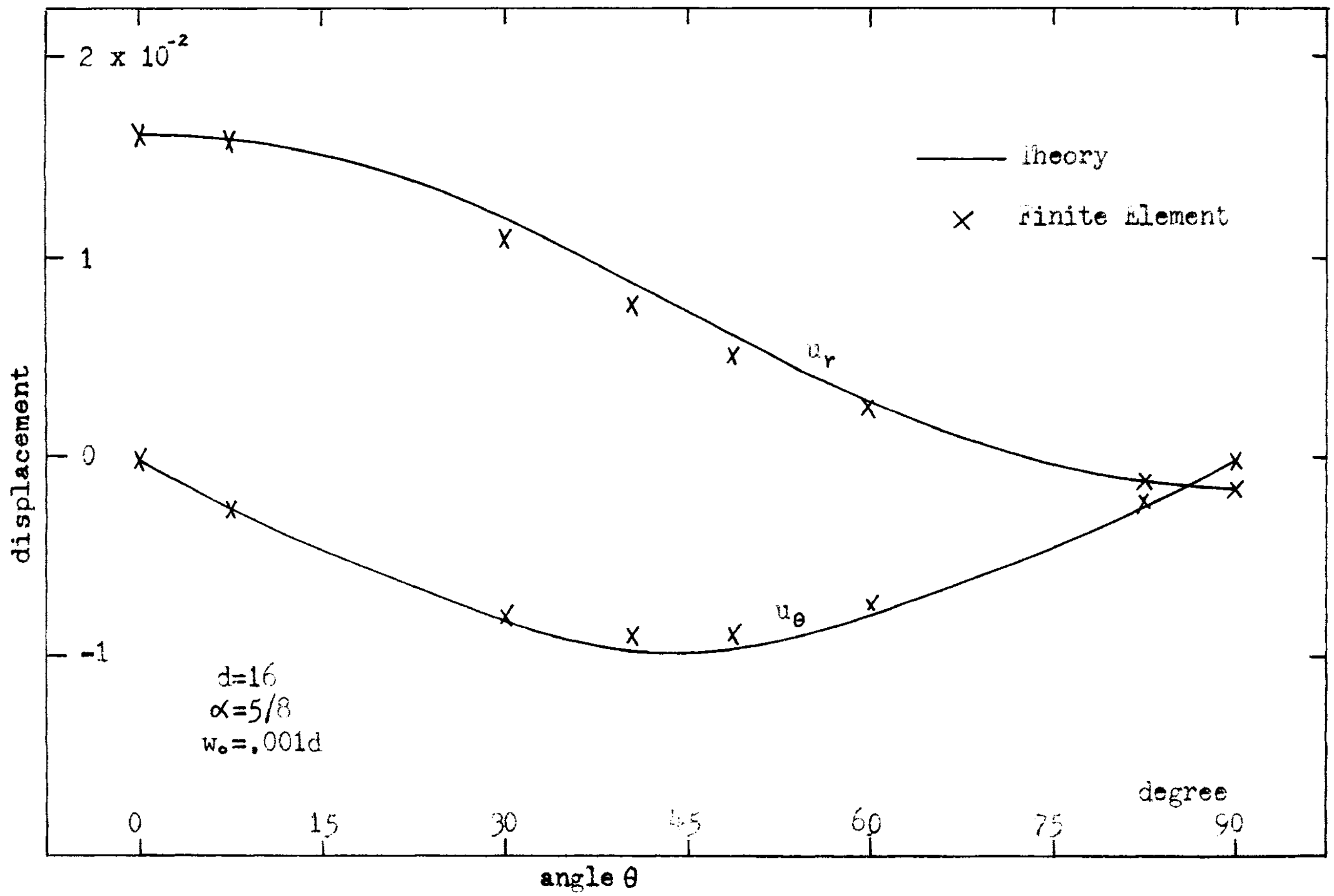


Figure A-4. Displacements at $r = d$ for $\beta = 10$ and $\gamma = 4$

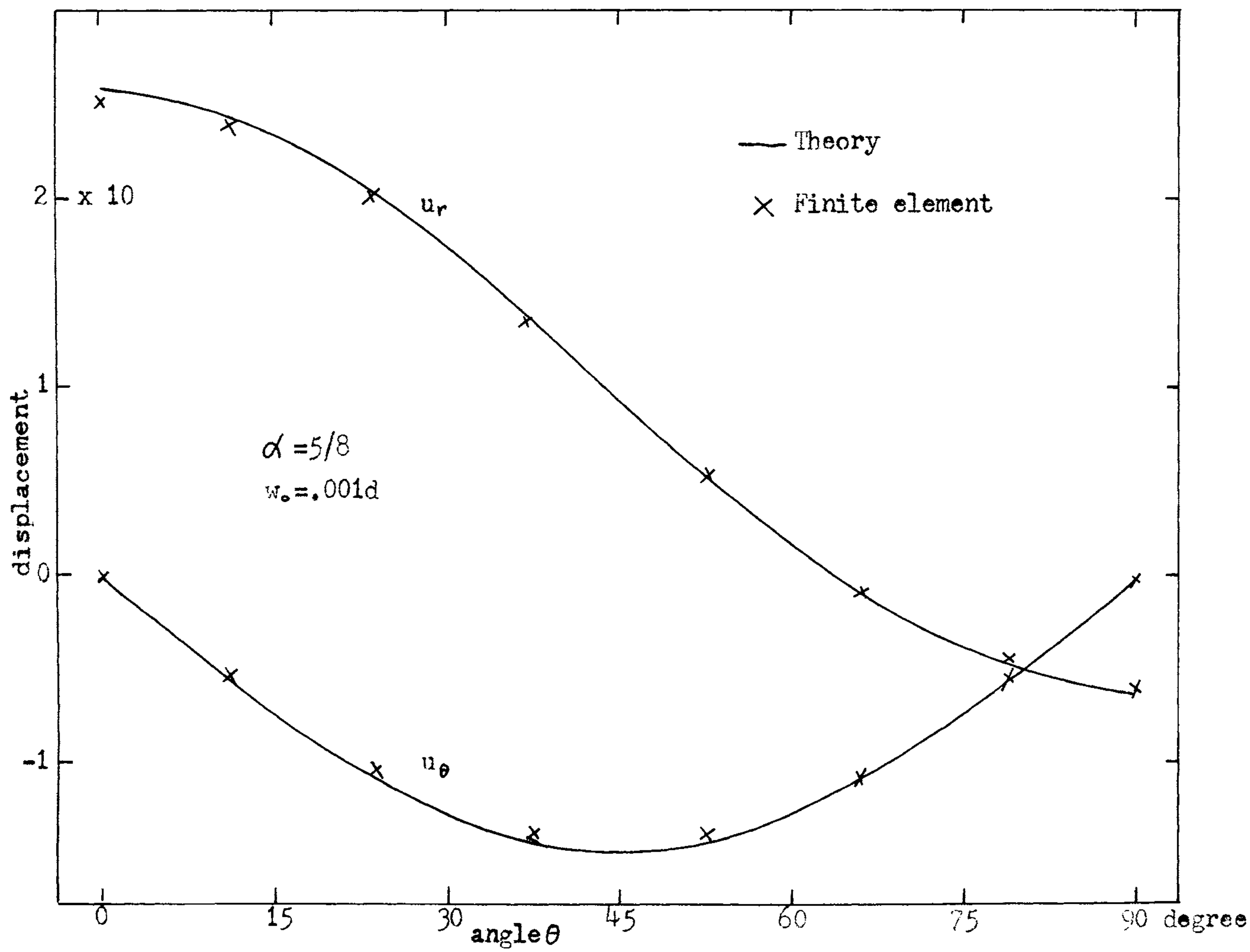


Figure A—5. Displacements at r = a for β=10 and γ=1

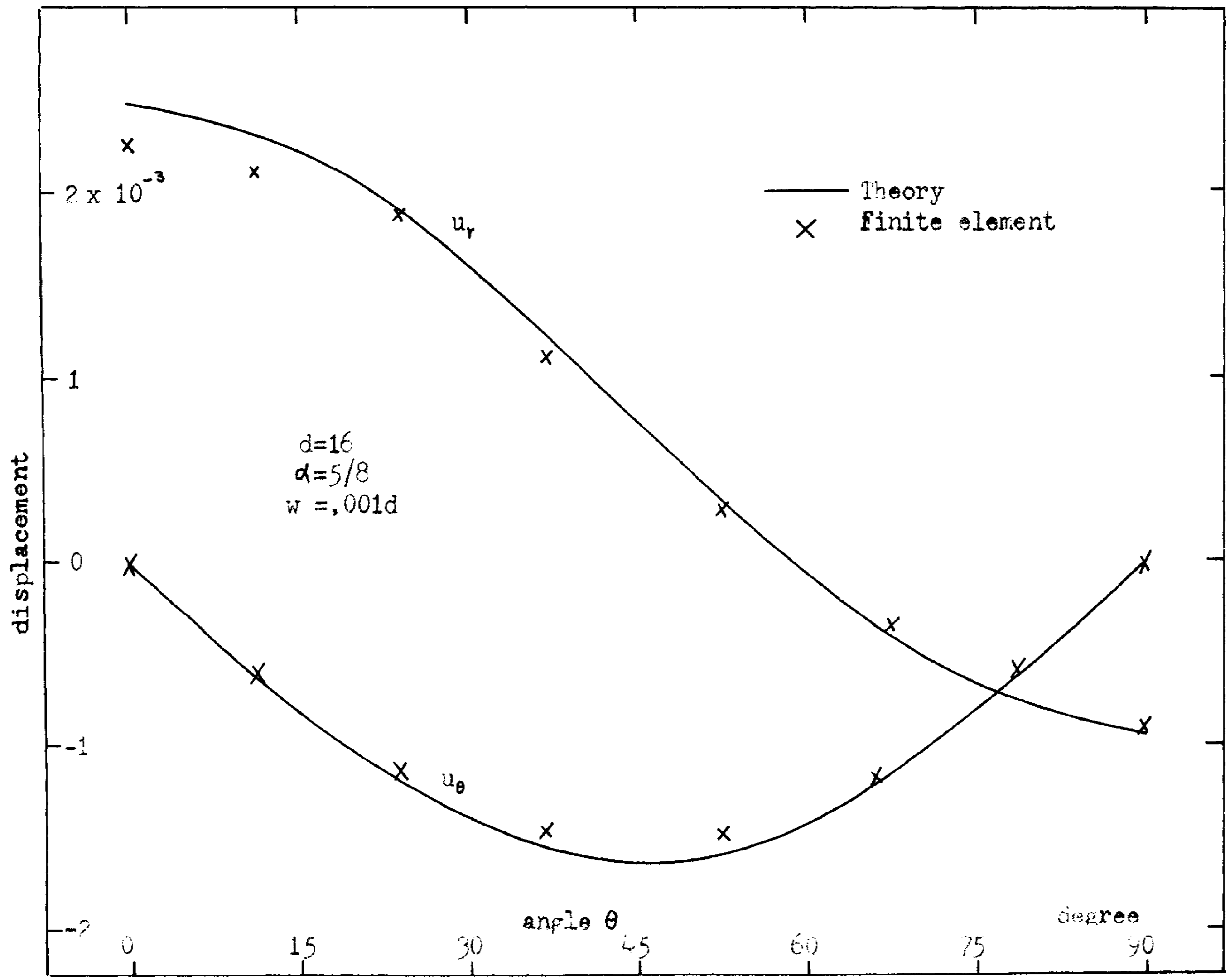


Figure A-6. Displacements at r = a for β=10 and γ=4

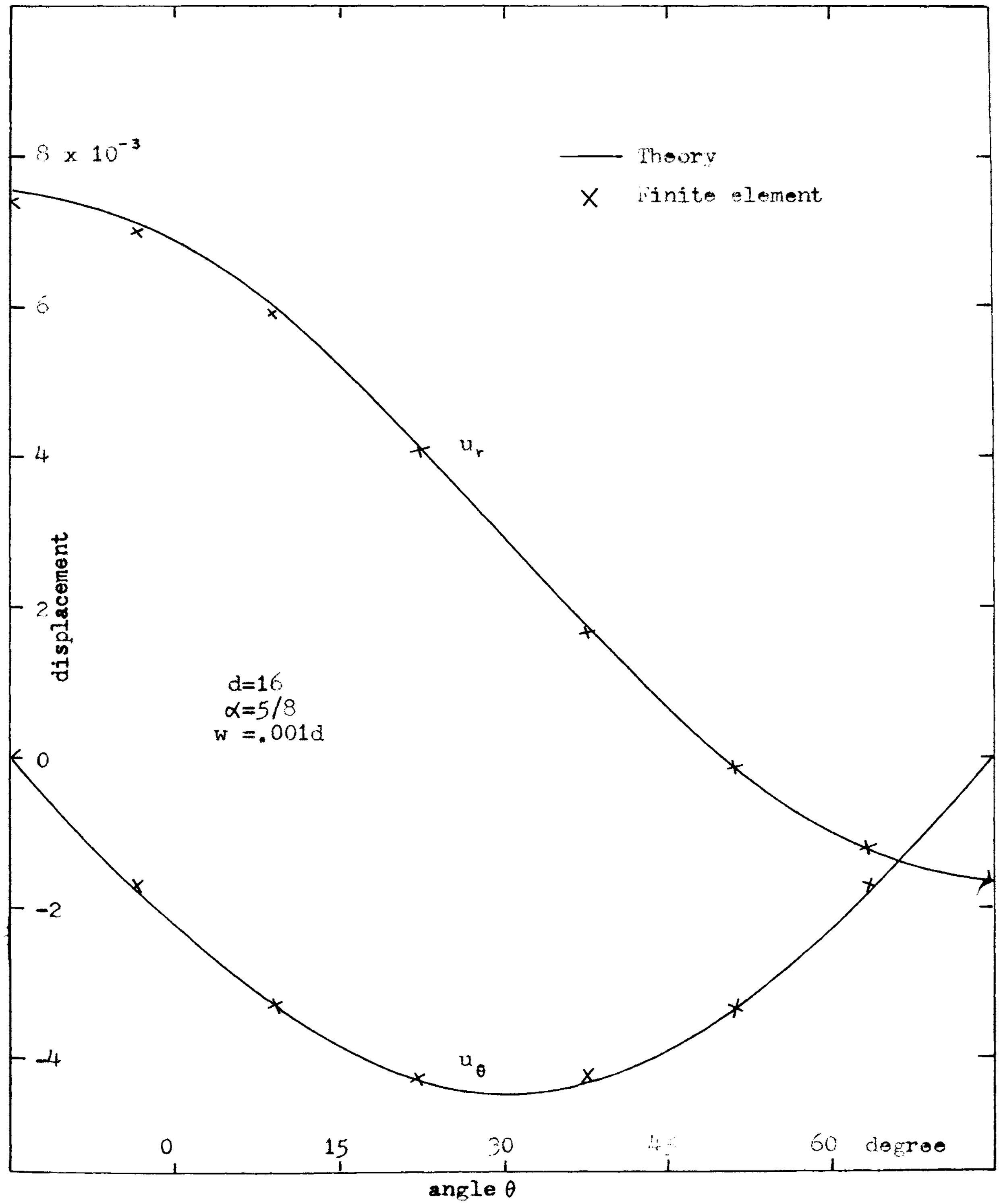
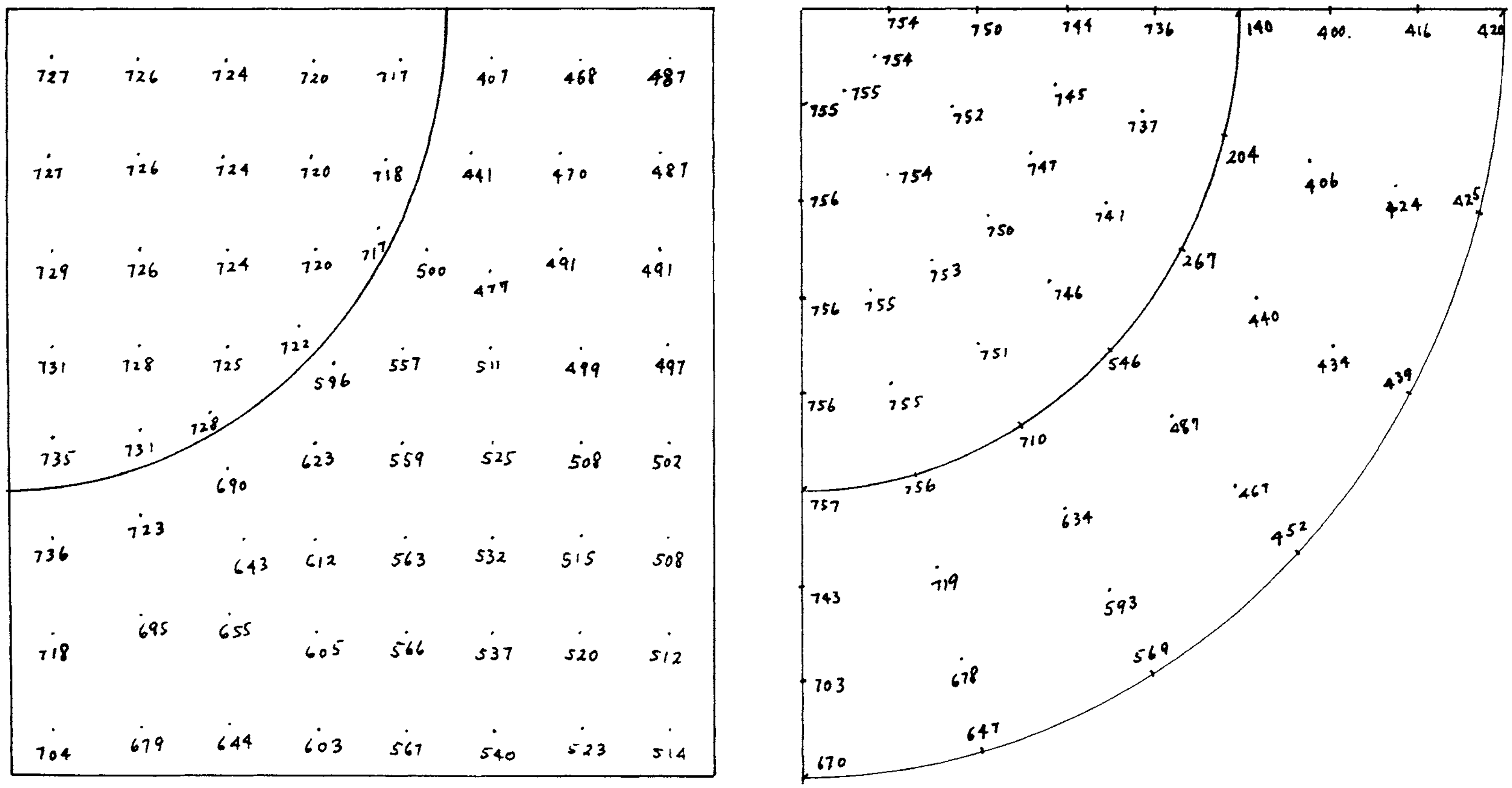


Figure A—7. Displacement at r = a for β =2 and γ=1



finite element

σ_{zz}

theory

Figure A-8. Vertical stresses for $\beta = 2$ and $\gamma = 1$

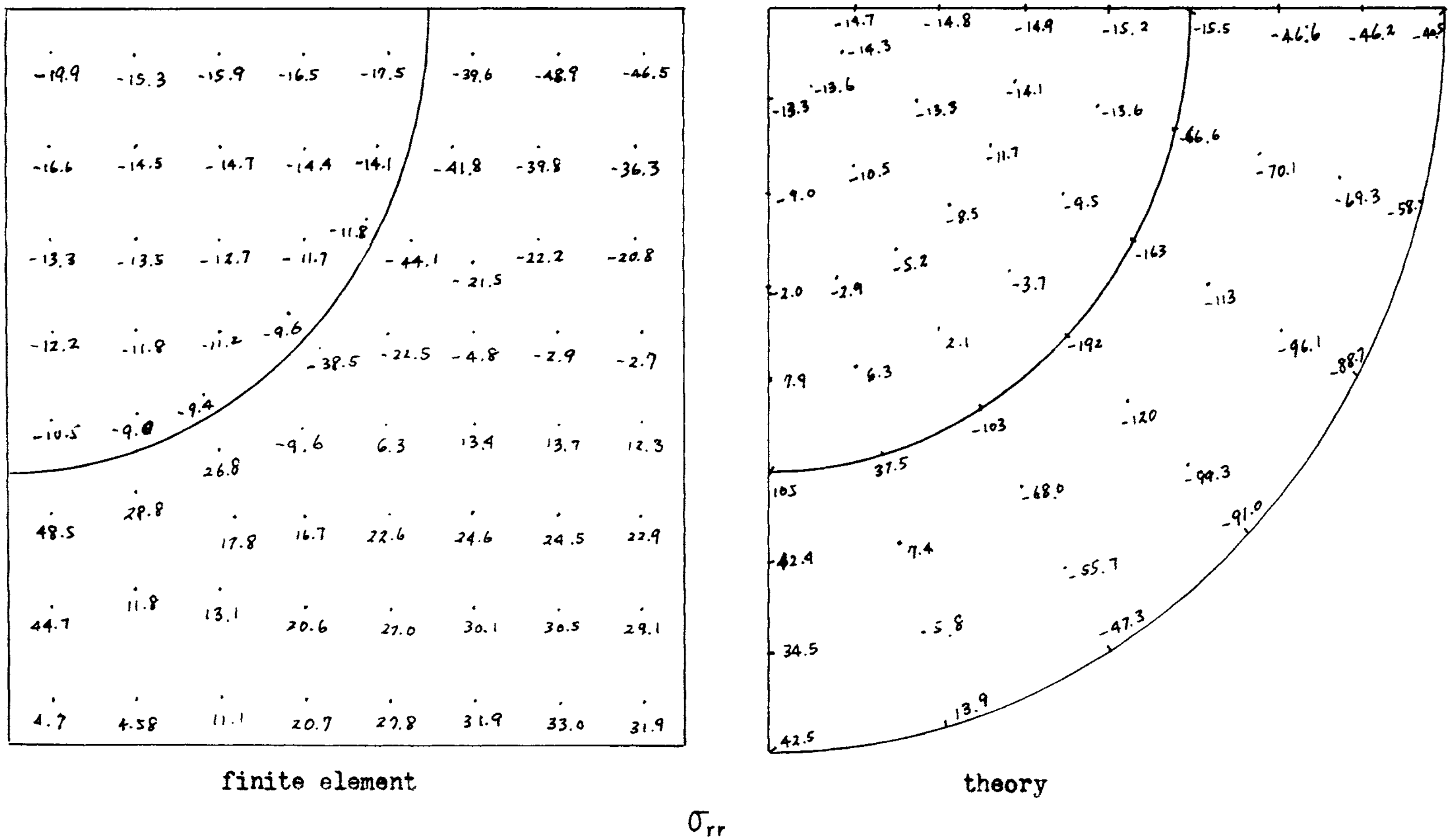
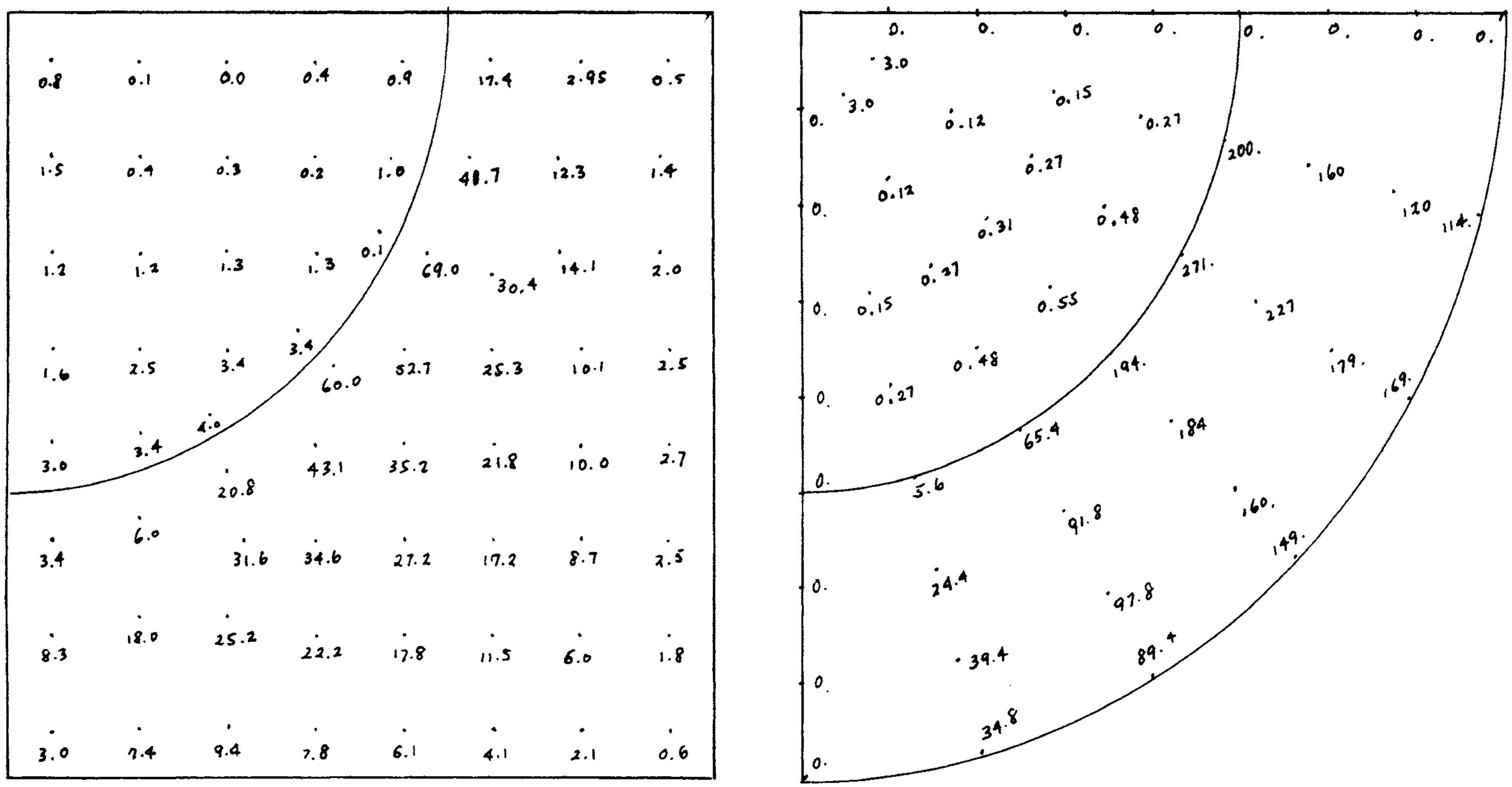


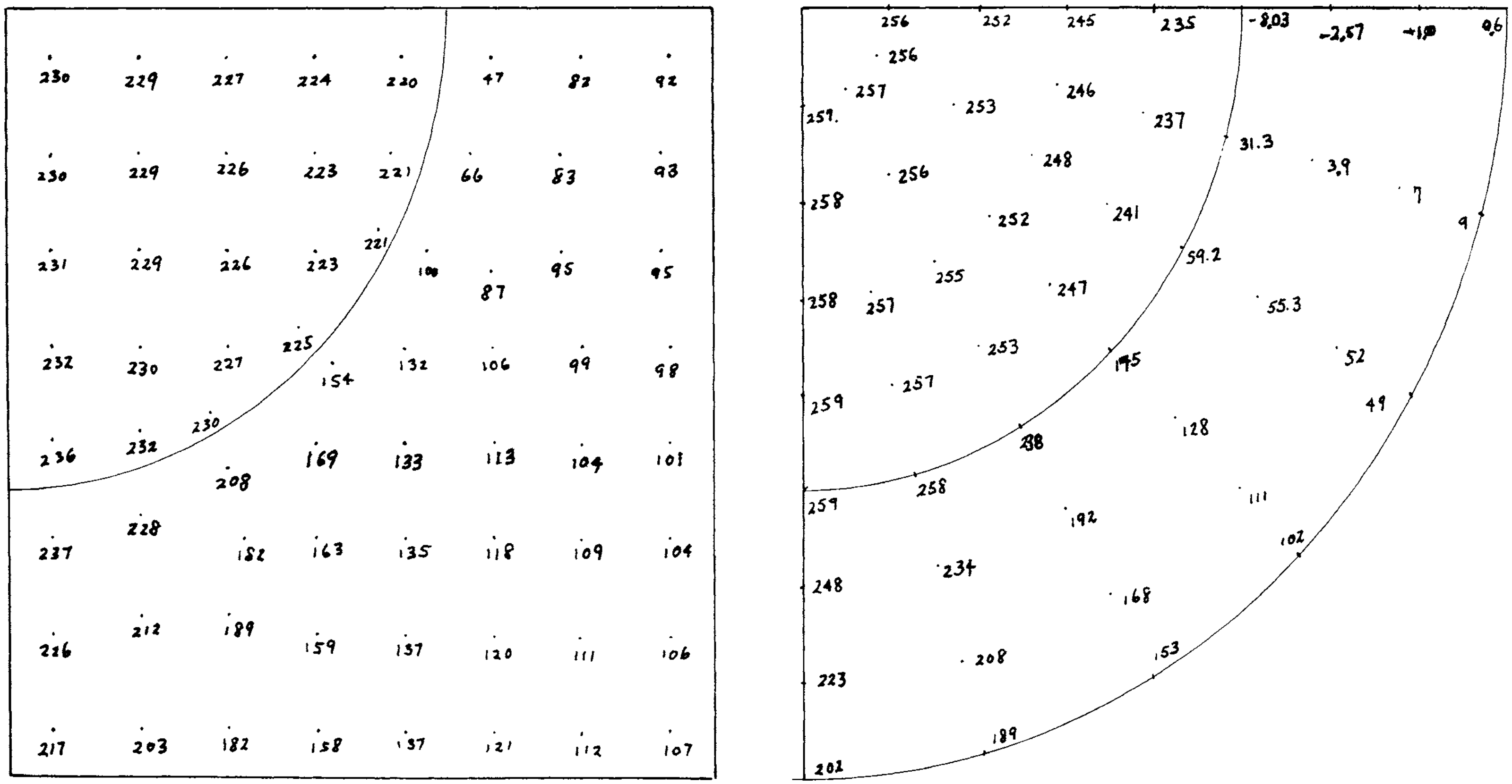
Figure A-9. Radial stresses for $\beta = 2$ and $\gamma = 1$



finite element theory

σ_{rz}

Figure A-10. Shear stresses for $\beta = 2$ and $\gamma = 1$

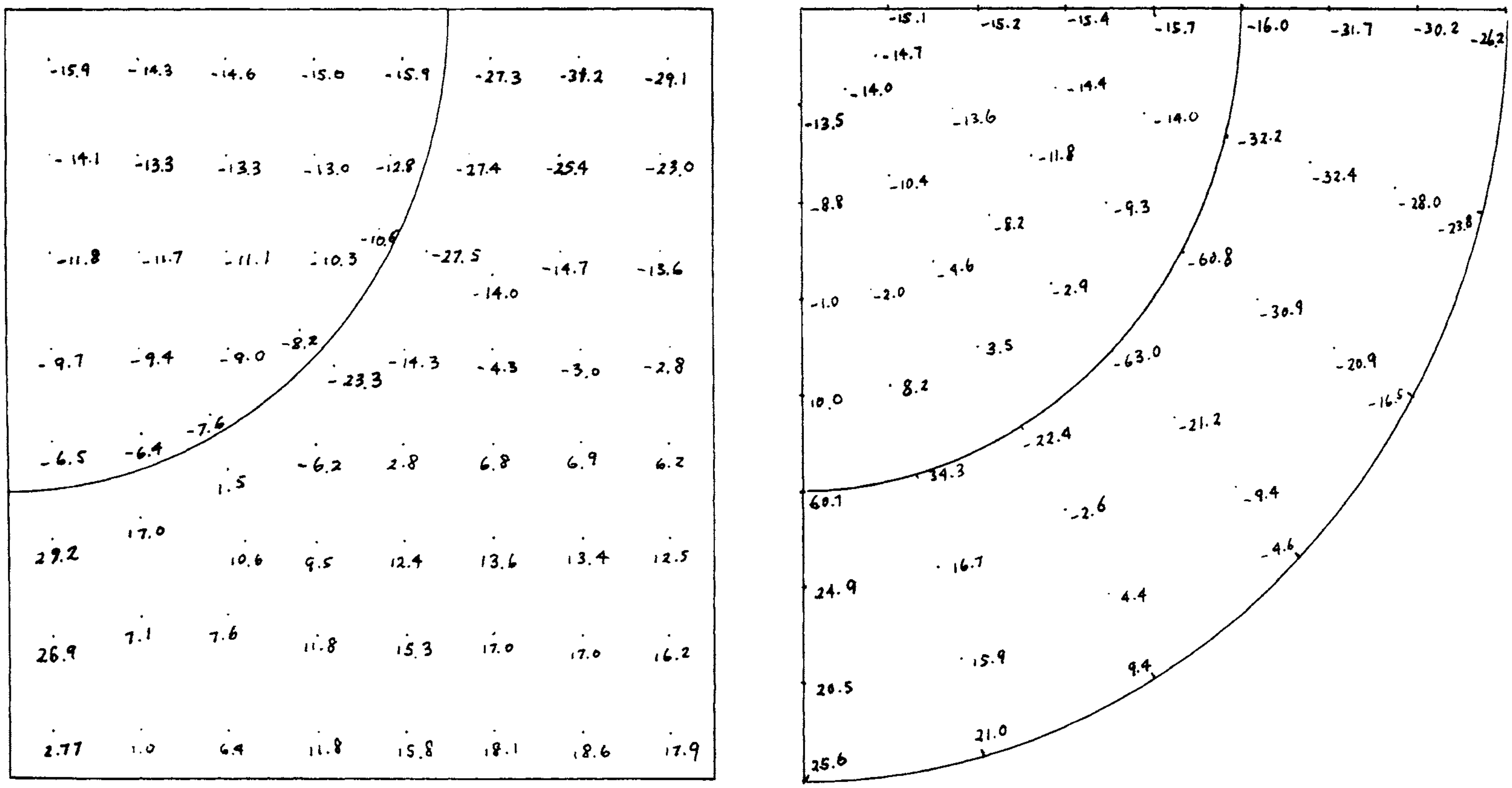


finite element

theory

σ_{zz}

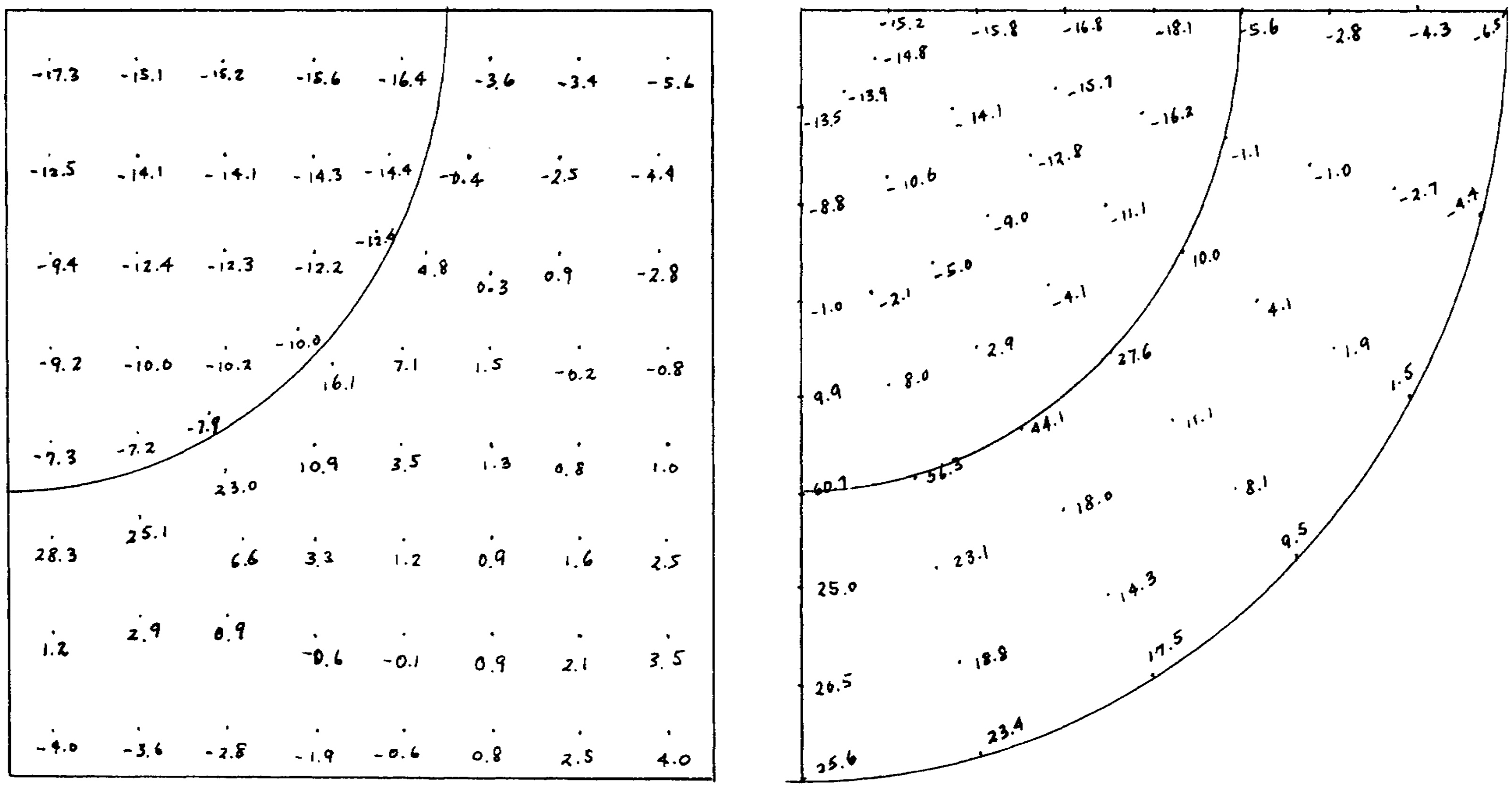
Figure A-11, Vertical stresses for β =10 and γ = 1



finite element   theory

σ_{rr}

Figure A-12. Radial stresses for $\beta = 10$ and $\gamma = 1$



finite element theory

$\sigma_{\theta\theta}$

Figure A-13. Tangential stresses for $\beta = 10$ and $\gamma = 1$

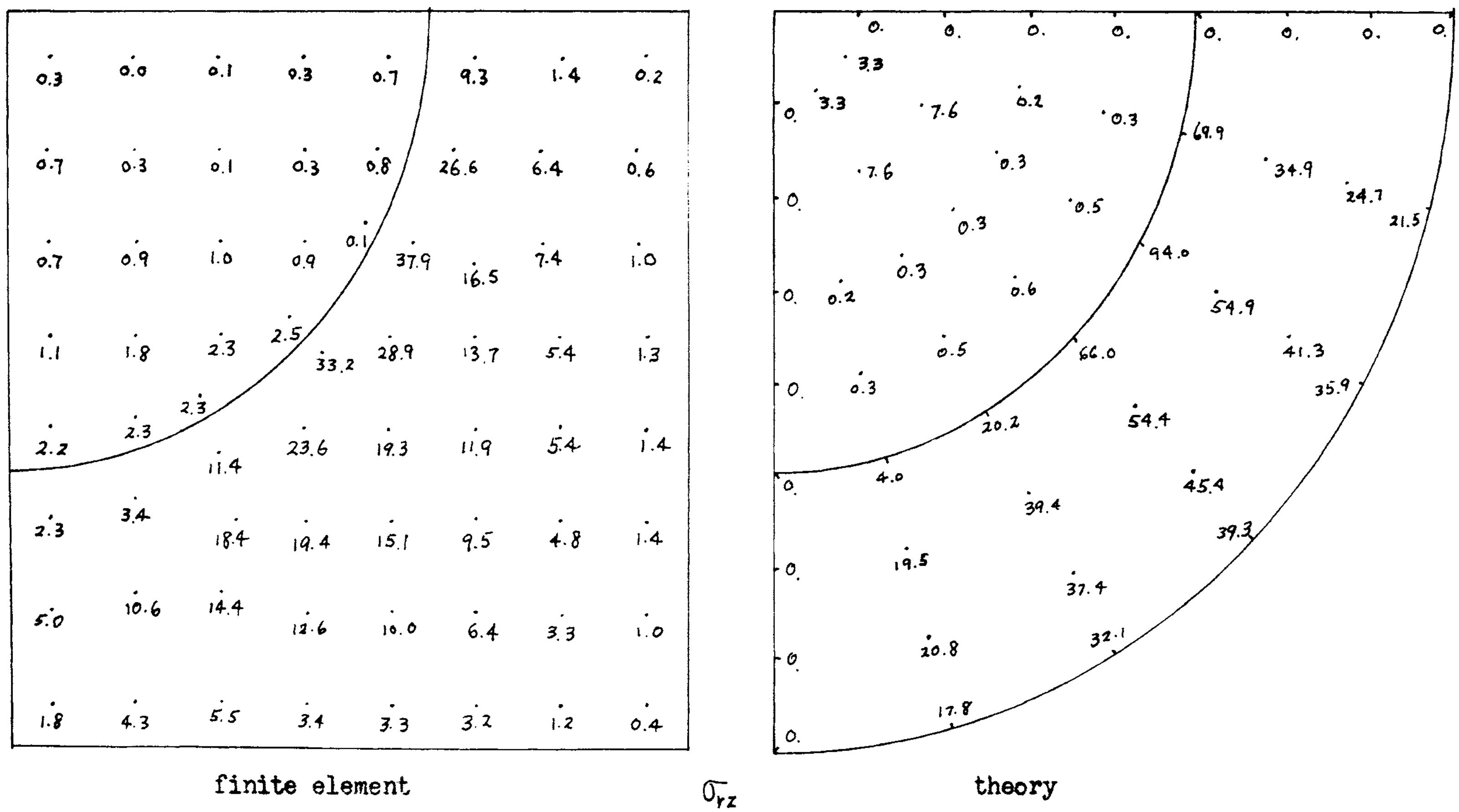


Figure A-14. Shear stresses for =10 and =1

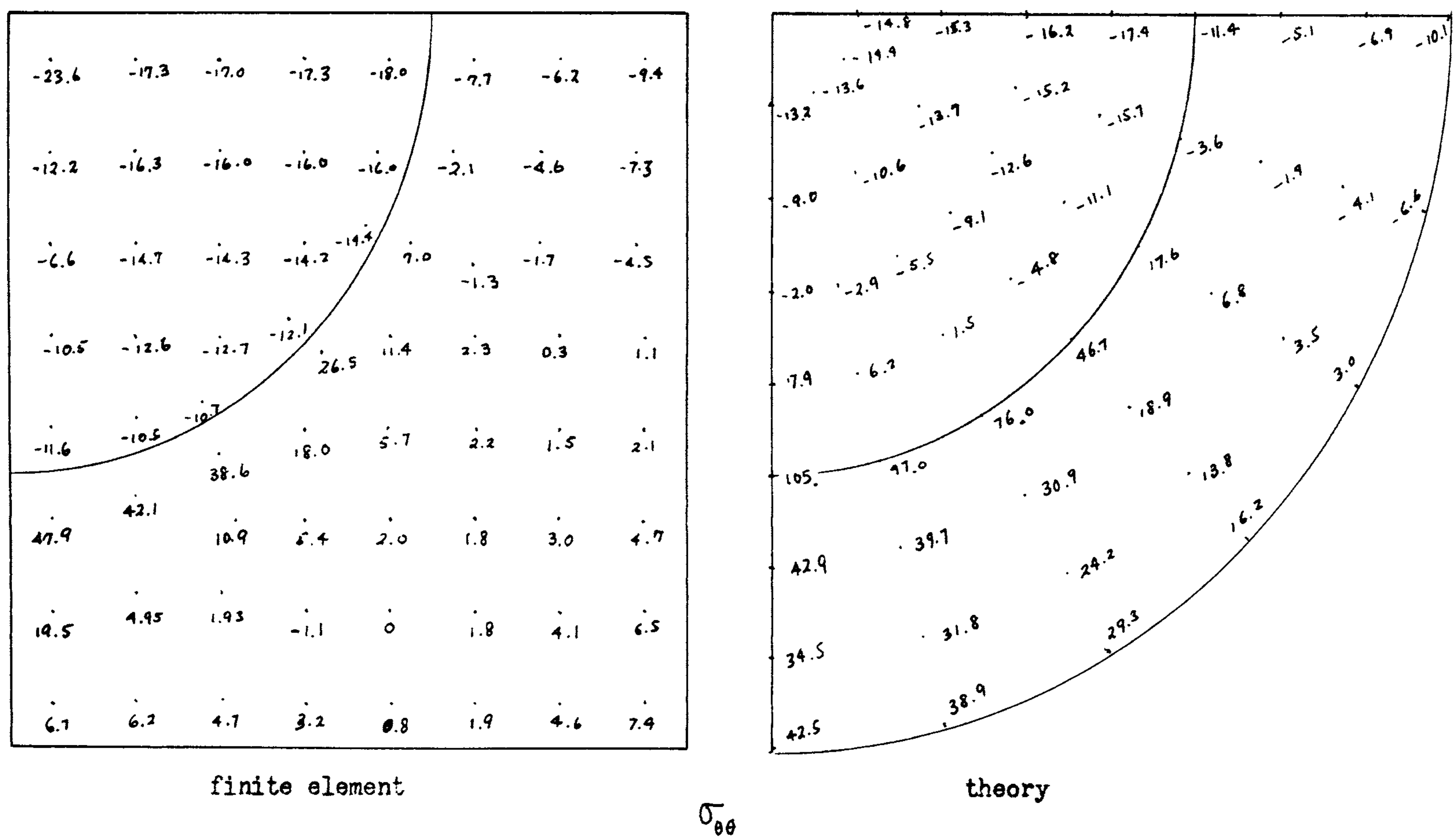


Figure A-15. Tangential stresses for $\beta=2$ and $\gamma=1$

APPENDIX B

STRESSES ALONG THE GRAIN BOUNDARY

Principal stresses, maximum stress differences, and induced stresses in principal directions in matrix are listed here for reference. All stresses were calculated on the basis of $E_1 = 100{,}000$ psi and $\nu_1 = .1$, and then divided by the apparent stress. The apparent stresses were calculated from $\sigma = w_o\, E/d$ and E was calculated from equation 4-15 when possible. Because of the truncation error by the computer, when $\alpha \geq .75$, the value of E becomes negative, depending on the value of β. For such cases, E was calculated by using equation 4-4. Values are listed for θ ranging from 0 to 90 degrees in 15 degree increments.

The effective Poisson's ratio, ν, was calculated from equation 4-7, which gives the upper bound. Because of this, the calculated values may be somewhat higher than actual values. The constants for solutions were directly calculated by solving the 9 by 9 matrix with the Gaussian elimination method. The listed are in the order of σ_1, σ_2, σ_3, $\sigma_1 - \sigma_3$, σ_{e1}, σ_{e2}, and σ_{e3}. The subscripts 1, 2, and 3 are given in the order of the absolute magnitudes of stresses, and the subscript e denotes induced stress.

BETA=.001

θ	σ_1	σ_2	σ_3	$\sigma_1 - \sigma_3$	σ_{e1}	σ_{e2}	σ_{e3}
ALPHA= 0.95							
0.	-0.3708E 00	-0.3708E 00	0.2523E 00	-0.6231E 00	-0.3589E 00	-0.3589E 00	0.3265E 00
15.	-0.3446E 00	-0.3382E 00	0.2431E 00	-0.5877E 00	-0.3350E 00	-0.3281E 00	0.3114E 00
30.	-0.2729E 00	-0.2506E 00	0.2193E 00	-0.4922E 00	-0.2698E 00	-0.2452E 00	0.2716E 00
45.	0.1929E 00	-0.1750E 00	-0.1370E 00	0.3298E 00	0.2241E 00	-0.1806E 00	-0.1388E 00
60.	0.1860E 00	-0.7707E-01	-0.4291E-01	0.2289E 00	0.1980E 00	-0.9138E-01	-0.5381E-01
75.	0.2111E 00	-0.5392E-02	-0.4194E-02	0.2153E 00	0.2121E 00	-0.2608E-01	-0.2477E-01
90.	0.2287E 00	0.2084E-01	0.1575E-02	0.2271E 00	0.2265E 00	-0.2186E-02	-0.2338E-01
ALPHA= 0.75							
0.	-0.3003E 00	-0.3003E 00	0.2795E 00	-0.5798E 00	-0.2982E 00	-0.2982E 00	0.3396E 00
15.	-0.2806E 00	0.2716E 00	-0.2585E 00	-0.2208E-01	-0.2819E 00	0.3255E 00	-0.2576E 00
30.	0.2590E 00	-0.2269E 00	-0.1535E 00	0.4125E 00	0.2971E 00	-0.2375E 00	-0.1567E 00
45.	0.2837E 00	-0.1535E 00	-0.5195E-01	0.3357E 00	0.3043E 00	-0.1767E 00	-0.6496E-01
60.	0.3813E 00	-0.8018E-01	-0.2322E-01	0.4045E 00	0.3916E 00	-0.1160E 00	-0.5333E-01
75.	0.4849E 00	-0.3439E-01	-0.2648E-01	0.5114E 00	0.4910E 00	-0.8023E-01	-0.7153E-01
90.	0.5263E 00	-0.4200E-01	-0.6820E-02	0.5332E 00	0.5312E 00	-0.9395E-01	-0.5525E-01
ALPHA= 0.55							
0.	-0.2684E 00	-0.2684E 00	0.2131E 00	-0.4815E 00	-0.2628E 00	-0.2628E 00	0.2668E 00
15.	-0.2535E 00	-0.2138E 00	0.2061E 00	-0.4595E 00	-0.2527E 00	-0.2091E 00	0.2528E 00
30.	-0.2127E 00	0.2002E 00	-0.7800E-01	-0.1347E 00	-0.2249E 00	0.2293E 00	-0.7676E-01
45.	0.2796E 00	-0.1570E 00	0.2011E-01	0.2595E 00	0.2933E 00	-0.1870E 00	0.7854E-02
60.	0.4631E 00	-0.1014E 00	0.1412E-01	0.4490E 00	0.4719E 00	-0.1491E 00	-0.2205E-01
75.	0.6164E 00	-0.6063E-01	-0.9209E-02	0.6256E 00	0.6234E 00	-0.1214E 00	-0.6479E-01
90.	0.6739E 00	-0.4572E-01	-0.1910E-01	0.6930E 00	0.6804E 00	-0.1112E 00	-0.8192E-01
ALPHA= 0.35							
0.	-0.2721E 00	-0.2721E 00	0.1854E 00	-0.4575E 00	-0.2635E 00	-0.2635E 00	0.2398E 00
15.	-0.2591E 00	-0.2092E 00	0.1786E 00	-0.4377E 00	-0.2560E 00	-0.2011E 00	0.2254E 00
30.	-0.2235E 00	0.1748E 00	-0.5197E-01	-0.1715E 00	-0.2358E 00	0.2024E 00	-0.4710E-01
45.	0.2922E 00	-0.1748E 00	0.4031E-01	0.2519E 00	0.3056E 00	-0.2081E 00	0.2858E-01
60.	0.5187E 00	-0.1262E 00	0.2342E-01	0.4953E 00	0.5290E 00	-0.1804E 00	-0.1583E-01
75.	0.6964E 00	-0.9058E-01	-0.8293E-03	0.6972E 00	0.7055E 00	-0.1601E 00	-0.6141E-01
90.	0.7622E 00	-0.7755E-01	-0.1047E-01	0.7727E 00	0.7710E 00	-0.1527E 00	-0.7894E-01
ALPHA= 0.15							
0.	-0.2765E 00	-0.2765E 00	0.1785E 00	-0.4551E 00	-0.2667E 00	-0.2667E 00	0.2338E 00
15.	-0.2639E 00	-0.2112E 00	0.1718E 00	-0.4357E 00	-0.2600E 00	-0.2020E 00	0.2193E 00
30.	-0.2295E 00	0.1685E 00	-0.4789E-01	-0.1816E 00	-0.2415E 00	0.1963E 00	-0.4180E-01
45.	0.2967E 00	-0.1824E 00	0.4259E-01	0.2541E 00	0.3107E 00	-0.2163E 00	0.3116E-01
60.	0.5341E 00	-0.1353E 00	0.2388E-01	0.5102E 00	0.5452E 00	-0.1911E 00	-0.1599E-01
75.	0.7183E 00	-0.1008E 00	-0.2953E-03	0.7186E 00	0.7284E 00	-0.1726E 00	-0.6204E-01
90.	0.7864E 00	-0.8820E-01	-0.9813E-02	0.7962E 00	0.7962E 00	-0.1659E 00	-0.7963E-01

BETA = .1

ALPHA= 0.95

0.	0.1719E 01	0.1727E 00	0.1727E 00	0.1546E 01	0.1684E 01	-0.1639E-01	-0.1639E-01
15.	0.1704E 01	0.1605E 00	0.5280E-01	0.1651E 01	0.1682E 01	-0.1517E-01	-0.1336E 00
30.	0.1600E 01	-0.2115E 00	0.1270E 00	0.1473E 01	0.1608E 01	-0.3842E 00	-0.1182E-01
45.	0.1334E 01	-0.4492E 00	0.8126E-01	0.1253E 01	0.1371E 01	-0.5907E 00	-0.7240E-02
60.	0.9136E 00	-0.5318E 00	0.3552E-01	0.8781E 00	0.9633E 00	-0.6267E 00	-0.2664E-02
75.	0.4137E 00	-0.4001E 00	0.2039E-02	0.4116E 00	0.4535E 00	-0.4417E 00	0.6862E-03
90.	-0.1029E 00	-0.1836E-01	-0.1022E-01	-0.9271E-01	-0.1001E 00	-0.7047E-02	0.1912E-02

ALPHA= 0.75

0.	0.1701E 01	0.1816E 00	0.1816E 00	0.1519E 01	0.1665E 01	-0.6650E-02	-0.6650E-02
15.	0.1685E 01	0.1705E 00	0.4769E-01	0.1637E 01	0.1663E 01	-0.2764E-02	-0.1378E 00
30.	0.1571E 01	-0.2492E 00	0.1400E 00	0.1431E 01	0.1582E 01	-0.4203E 00	0.7853E-02
45.	0.1285E 01	-0.5249E 00	0.9841E-01	0.1187E 01	0.1328E 01	-0.6633E 00	0.2236E-01
60.	0.8422E 00	-0.6426E 00	0.5682E-01	0.7853E 00	0.9007E 00	-0.7325E 00	0.3686E-01
75.	-0.5416E 00	0.3305E 00	0.2637E-01	-0.5679E 00	-0.5773E 00	0.3820E 00	0.4748E-01
90.	-0.3330E 00	-0.2841E-01	0.1522E-01	-0.3482E 00	-0.3317E 00	0.3373E-02	0.5136E-01

ALPHA= 0.55

0.	0.1590E 01	0.1471E 00	0.1471E 00	0.1443E 01	0.1561E 01	-0.2666E-01	-0.2666E-01
15.	0.1583E 01	0.1386E 00	0.1231E-01	0.1570E 01	0.1568E 01	-0.2090E-01	-0.1598E 00
30.	0.1489E 01	-0.2831E 00	0.1154E 00	0.1374E 01	0.1506E 01	-0.4435E 00	-0.5159E-02
45.	0.1229E 01	-0.5543E 00	0.8382E-01	0.1145E 01	0.1276E 01	-0.6855E 00	0.1634E-01
60.	0.8135E 00	-0.6701E 00	0.5218E-01	0.7613E 00	0.8753E 00	-0.7566E 00	0.3784E-01
75.	-0.5806E 00	0.3351E 00	0.2902E-01	-0.6096E 00	-0.6170E 00	0.3902E 00	0.5357E-01
90.	-0.4194E 00	0.3151E-01	0.2055E-01	-0.4399E 00	-0.4246E 00	0.7139E-01	0.5933E-01

ALPHA= 0.35

0.	0.1547E 01	0.1350E 00	0.1350E 00	0.1412E 01	0.1520E 01	-0.3326E-01	-0.3326E-01
15.	0.1543E 01	0.1275E 00	-0.7300E-03	0.1543E 01	0.1530E 01	-0.2673E-01	-0.1677E 00
30.	0.1456E 01	-0.2971E 00	0.1070E 00	0.1349E 01	0.1475E 01	-0.4534E 00	-0.8908E-02
45.	0.1204E 01	-0.5690E 00	0.7897E-01	0.1125E 01	0.1253E 01	-0.6973E 00	0.1544E-01
60.	0.7994E 00	-0.6874E 00	0.5098E-01	0.7484E 00	0.8630E 00	-0.7725E 00	0.3979E-01
75.	-0.6070E 00	0.3358E 00	0.3049E-01	-0.6375E 00	-0.6437E 00	0.3934E 00	0.5762E-01
90.	-0.4657E 00	0.5418E-01	0.2299E-01	-0.4887E 00	-0.4734E 00	0.9846E-01	0.6414E-01

ALPHA= 0.15

0.	0.1534E 01	0.1335E 00	0.1335E 00	0.1401E 01	0.1507E 01	-0.3323E-01	-0.3323E-01
15.	0.1530E 01	0.1262E 00	-0.2253E-02	0.1532E 01	0.1517E 01	-0.2654E-01	-0.1679E 00
30.	0.1444E 01	-0.2989E 00	0.1062E 00	0.1338E 01	0.1463E 01	-0.4539E 00	-0.8261E-02
45.	0.1194E 01	-0.5717E 00	0.7892E-01	0.1115E 01	0.1243E 01	-0.6990E 00	0.1670E-01
60.	0.7919E 00	-0.6924E 00	0.5162E-01	0.7403E 00	0.8560E 00	-0.7768E 00	0.4167E-01
75.	-0.6169E 00	0.3337E 00	0.3163E-01	-0.6485E 00	-0.6534E 00	0.3922E 00	0.5994E-01
90.	-0.4830E 00	0.5976E-01	0.2431E-01	-0.5073E 00	-0.4914E 00	0.1056E 00	0.6663E-01

BETA= 3.0

ALPHA= 0.95

0.	0.2960E 01	0.3628E 00	0.3628E 00	0.2597E 01	0.2888E 01	0.3055E-01	0.3055E-01
15.	0.2886E 01	0.3369E 00	0.1943E 00	0.2691E 01	0.2832E 01	0.2896E-01	-0.1280E 00
30.	0.2614E 01	0.2662E 00	-0.1981E 00	0.2812E 01	0.2607E 01	0.2462E-01	-0.4861E 00
45.	0.2093E 01	-0.5848E 00	0.1695E 00	0.1924E 01	0.2135E 01	-0.8111E 00	0.1869E-01
60.	0.1366E 01	-0.7648E 00	0.7288E-01	0.1293E 01	0.1435E 01	-0.9087E 00	0.1276E-01
75.	-0.6254E 00	0.5625E 00	0.2125E-02	-0.6275E 00	-0.6819E 00	0.6249E 00	0.8413E-02
90.	-0.2072E 00	-0.9880E-01	-0.2377E-01	-0.1834E 00	-0.1949E 00	-0.7571E-01	0.6824E-02

ALPHA= 0.75

0.	0.2330E 01	0.2695E 00	0.2695E 00	0.2060E 01	0.2276E 01	0.9628E-02	0.9628E-02
15.	0.2281E 01	0.2532E 00	0.1042E 00	0.2177E 01	0.2245E 01	0.1469E-01	-0.1492E 00
30.	0.2077E 01	-0.2754E 00	0.2087E 00	0.1869E 01	0.2084E 01	-0.5040E 00	0.2852E-01
45.	0.1654E 01	-0.6489E 00	0.1479E 00	0.1506E 01	0.1704E 01	-0.8290E 00	0.4741E-01
60.	0.1045E 01	-0.8371E 00	0.8706E-01	0.9576E 00	0.1120E 01	-0.9502E 00	0.6631E-01
75.	-0.7547E 00	0.3787E 00	0.4254E-01	-0.7972E 00	-0.7968E 00	0.4499E 00	0.8014E-01
90.	-0.5469E 00	-0.4273E-01	0.2624E-01	-0.5731E 00	-0.5452E 00	0.9331E-02	0.8520E-01

ALPHA= 0.55

0.	0.2035E 01	0.2221E 00	0.2221E 00	0.1813E 01	0.1991E 01	-0.3601E-02	-0.3602E-02
15.	0.2017E 01	0.2089E 00	0.4724E-01	0.1970E 01	0.1992E 01	0.2440E-02	-0.1754E 00
30.	0.1880E 01	-0.3420E 00	0.1727E 00	0.1707E 01	0.1897E 01	-0.5473E 00	0.1895E-01
45.	0.1531E 01	-0.7131E 00	0.1233E 00	0.1408E 01	0.1590E 01	-0.8785E 00	0.4150E-01
60.	0.9970E 00	-0.8980E 00	0.7396E-01	0.9231E 00	0.1079E 01	-0.1005E 01	0.6405E-01
75.	-0.8364E 00	0.4089E 00	0.3780E-01	-0.8742E 00	-0.8811E 00	0.4887E 00	0.8056E-01
90.	-0.6963E 00	0.7600E-01	0.2457E-01	-0.7209E 00	-0.7064E 00	0.1432E 00	0.8660E-01

ALPHA= 0.35

0.	0.1927E 01	0.2057E 00	0.2057E 00	0.1722E 01	0.1886E 01	-0.7589E-02	-0.7590E-02
15.	0.1919E 01	0.1936E 00	0.2694E-01	0.1892E 01	0.1897E 01	-0.1052E-02	-0.1843E 00
30.	0.1803E 01	-0.3670E 00	0.1604E 00	0.1642E 01	0.1823E 01	-0.5633E 00	0.1681E-01
45.	0.1479E 01	-0.7406E 00	0.1151E 00	0.1364E 01	0.1542E 01	-0.9000E 00	0.4121E-01
60.	0.9722E 00	-0.9308E 00	0.6975E-01	0.9025E 00	0.1058E 01	-0.1035E 01	0.6560E-01
75.	-0.8860E 00	0.4171E 00	0.3657E-01	-0.9226E 00	-0.9314E 00	0.5020E 00	0.8347E-01
90.	-0.7722E 00	0.1165E 00	0.2443E-01	-0.7966E 00	-0.7863E 00	0.1913E 00	0.9000E-01

ALPHA= 0.15

0.	0.1894E 01	0.2019E 00	0.2019E 00	0.1692E 01	0.1853E 01	-0.7645E-02	-0.7645E-02
15.	0.1887E 01	0.1901E 00	0.2353E-01	0.1863E 01	0.1866E 01	-0.9581E-03	-0.1842E 00
30.	0.1774E 01	-0.3695E 00	0.1577E 00	0.1616E 01	0.1795E 01	-0.5626E 00	0.1731E-01
45.	0.1456E 01	-0.7431E 00	0.1135E 00	0.1342E 01	0.1519E 01	-0.9000E 00	0.4227E-01
60.	0.9573E 00	-0.9365E 00	0.6931E-01	0.8880E 00	0.1044E 01	-0.1039E 01	0.6722E-01
75.	-0.9002E 00	0.4147E 00	0.3694E-01	-0.9372E 00	-0.9454E 00	0.5010E 00	0.8549E-01
90.	-0.7963E 00	0.1255E 00	0.2510E-01	-0.8214E 00	-0.8114E 00	0.2026E 00	0.9218E-01

BETA = 10.0

ALPHA= 0.95

0.	0.4270E 01	0.5092E 00	0.5092E 00	0.3761E 01	0.4169E 01	0.3127E-01	0.3127E-01
15.	0.4102E 01	0.4726E 00	0.3222E 00	0.3780E 01	0.4023E 01	0.3016E-01	-0.1353E 00
30.	0.3589E 01	0.3725E 00	-0.1357E 00	0.3725E 01	0.3566E 01	0.2713E-01	-0.5319E 00
45.	0.2754E 01	-0.6265E 00	0.2358E 00	0.2519E 01	0.2793E 01	-0.9255E 00	0.2298E-01
60.	0.1705E 01	-0.9034E 00	0.9901E-01	0.1606E 01	0.1786E 01	-0.1084E 01	0.1883E-01
75.	-0.7982E 00	0.6292E 00	-0.1098E-02	-0.7971E 00	-0.8610E 00	0.7092E 00	0.1579E-01
90.	-0.2779E 00	-0.2464E 00	-0.3774E-01	-0.2401E 00	-0.2494E 00	-0.2148E 00	0.1468E-01

ALPHA= 0.75

0.	0.2576E 01	0.2717E 00	0.2717E 00	0.2304E 01	0.2521E 01	-0.1303E-01	-0.1303E-01
15.	0.2507E 01	0.2552E 00	0.1077E 00	0.2400E 01	0.2471E 01	-0.6297E-02	-0.1685E 00
30.	0.2257E 01	-0.2761E 00	0.2101E 00	0.2046E 01	0.2263E 01	-0.5228E 00	0.1210E-01
45.	0.1777E 01	-0.6635E 00	0.1485E 00	0.1628E 01	0.1828E 01	-0.8560E 00	0.3722E-01
60.	0.1114E 01	-0.8681E 00	0.8696E-01	0.1027E 01	0.1192E 01	-0.9883E 00	0.6234E-01
75.	-0.7976E 00	0.4091E 00	0.4188E-01	-0.8395E 00	-0.8427E 00	0.4846E 00	0.8073E-01
90.	-0.6023E 00	0.2539E-01	-0.1862E-01	-0.5836E 00	-0.6029E 00	0.8747E-01	0.3907E-01

ALPHA= 0.55

0.	0.2213E 01	0.2400E 00	0.2400E 00	0.1973E 01	0.2165E 01	-0.5361E-02	-0.5361E-02
15.	0.2189E 01	0.2254E 00	0.5377E-01	0.2135E 01	0.2161E 01	0.1113E-02	-0.1876E 00
30.	0.2031E 01	-0.3641E 00	0.1855E 00	0.1845E 01	0.2049E 01	-0.5857E 00	0.1880E-01
45.	0.1648E 01	-0.7677E 00	0.1310E 00	0.1517E 01	0.1711E 01	-0.9456E 00	0.4296E-01
60.	0.1070E 01	-0.9770E 00	0.7644E-01	0.9937E 00	0.1160E 01	-0.1092E 01	0.6713E-01
75.	-0.9277E 00	0.4449E 00	0.3653E-01	-0.9643E 00	-0.9759E 00	0.5341E 00	0.8481E-01
90.	-0.7959E 00	0.1023E 00	0.2193E-01	-0.8178E 00	-0.8083E 00	0.1796E 00	0.9129E-01

ALPHA= 0.35

0.	0.2075E 01	0.2276E 00	0.2276E 00	0.1847E 01	0.2030E 01	-0.2654E-02	-0.2655E-02
15.	0.2065E 01	0.2138E 00	0.3394E-01	0.2031E 01	0.2040E 01	0.3929E-02	-0.1939E 00
30.	0.1936E 01	-0.3945E 00	0.1761E 00	0.1760E 01	0.1958E 01	-0.6058E 00	0.2192E-01
45.	0.1586E 01	-0.8049E 00	0.1246E 00	0.1461E 01	0.1654E 01	-0.9759E 00	0.4649E-01
60.	0.1042E 01	-0.1022E 01	0.7306E-01	0.9690E 00	0.1137E 01	-0.1134E 01	0.7106E-01
75.	-0.9922E 00	0.4551E 00	0.3535E-01	-0.1028E 01	-0.1041E 01	0.5508E 00	0.8905E-01
90.	-0.8874E 00	0.1465E 00	0.2154E-01	-0.9090E 00	-0.9042E 00	0.2331E 00	0.9563E-01

ALPHA= 0.15

0.	0.2031E 01	0.2234E 00	0.2234E 00	0.1808E 01	0.1987E 01	-0.2093E-02	-0.2095E-02
15.	0.2023E 01	0.2099E 00	0.3010E-01	0.1993E 01	0.1999E 01	0.4592E-02	-0.1932E 00
30.	0.1899E 01	-0.3972E 00	0.1730E 00	0.1726E 01	0.1921E 01	-0.6044E 00	0.2286E-01
45.	0.1556E 01	-0.8074E 00	0.1227E 00	0.1433E 01	0.1625E 01	-0.9752E 00	0.4782E-01
60.	-0.1028E 01	0.1024E 01	0.7234E-01	-0.1100E 01	-0.1138E 01	0.1119E 01	0.7277E-01
75.	-0.1008E 01	0.4522E 00	0.3548E-01	-0.1043E 01	-0.1057E 01	0.5494E 00	0.9104E-01
90.	-0.9136E 00	0.1562E 00	0.2199E-01	-0.9356E 00	-0.9314E 00	0.2454E 00	0.9773E-01

BETA = 30.0

ALPHA= 0.95							
0.	0.5279E 01	0.6036E 00	0.6036E 00	0.4675E 01	0.5158E 01	0.1537E-01	0.1537E-01
15.	0.5029E 01	0.5599E 00	0.4119E 00	0.4617E 01	0.4931E 01	0.1585E-01	-0.1469E 00
30.	0.4304E 01	0.4404E 00	-0.7188E-01	0.4376E 01	0.4267E 01	0.1716E-01	-0.5464E 00
45.	0.3202E 01	-0.6198E 00	0.2772E 00	0.2925E 01	0.3236E 01	-0.9677E 00	0.1895E-01
60.	0.1900E 01	-0.9680E 00	0.1139E 00	0.1786E 01	0.1985E 01	-0.1169E 01	0.2073E-01
75.	-0.9109E 00	0.6346E 00	-0.5582E-02	-0.9053E 00	-0.9738E 00	0.7263E 00	0.2204E-01
90.	-0.3993E 00	-0.3191E 00	-0.4932E-01	-0.3500E 00	-0.3625E 00	-0.2743E 00	0.2252E-01
ALPHA= 0.75							
0.	0.2665E 01	0.2643E 00	0.2643E 00	0.2401E 01	0.2612E 01	-0.2862E-01	-0.2862E-01
15.	0.2588E 01	0.2482E 00	0.1040E 00	0.2484E 01	0.2553E 01	-0.2103E-01	-0.1796E 00
30.	0.2318E 01	-0.2744E 00	0.2040E 00	0.2114E 01	0.2325E 01	-0.5266E 00	-0.2801E-03
45.	0.1817E 01	-0.6604E 00	0.1438E 00	0.1674E 01	0.1869E 01	-0.8565E 00	0.2806E-01
60.	0.1139E 01	-0.8679E 00	0.8350E-01	0.1055E 01	0.1217E 01	-0.9901E 00	0.5640E-01
75.	-0.8028E 00	0.4251E 00	0.3938E-01	-0.8422E 00	-0.8492E 00	0.5015E 00	0.7715E-01
90.	-0.6158E 00	0.2323E-01	0.7421E-03	-0.6166E 00	-0.6182E 00	0.8474E-01	0.6000E-01
ALPHA= 0.55							
0.	0.2283E 01	0.2445E 00	0.2445E 00	0.2039E 01	0.2234E 01	-0.8293E-02	-0.8294E-02
15.	0.2255E 01	0.2294E 00	0.5476E-01	0.2201E 01	0.2227E 01	-0.1588E-02	-0.1937E 00
30.	0.2089E 01	-0.3724E 00	0.1884E 00	0.1900E 01	0.2107E 01	-0.6001E 00	0.1673E-01
45.	0.1692E 01	-0.7873E 00	0.1322E 00	0.1560E 01	0.1757E 01	-0.9697E 00	0.4176E-01
60.	0.1099E 01	-0.1005E 01	0.7611E-01	0.1023E 01	0.1192E 01	-0.1123E 01	0.6679E-01
75.	-0.9610E 00	0.4603E 00	0.3503E-01	-0.9961E 00	-0.1011E 01	0.5529E 00	0.8511E-01
90.	-0.8326E 00	0.1143E 00	0.1999E-01	-0.8526E 00	-0.8460E 00	0.1956E 00	0.9182E-01
ALPHA= 0.35							
0.	0.2133E 01	0.2352E 00	0.2352E 00	0.1897E 01	0.2086E 01	-0.1567E-02	-0.1568E-02
15.	0.2121E 01	0.2208E 00	0.3605E-01	0.2085E 01	0.2096E 01	0.5042E-02	-0.1982E 00
30.	0.1988E 01	-0.4053E 00	0.1814E 00	0.1807E 01	0.2010E 01	-0.6222E 00	0.2310E-01
45.	0.1627E 01	-0.8295E 00	0.1275E 00	0.1500E 01	0.1697E 01	-0.1005E 01	0.4777E-01
60.	0.1070E 01	-0.1057E 01	0.7370E-01	0.9960E 00	0.1168E 01	-0.1171E 01	0.7244E-01
75.	-0.1033E 01	0.4708E 00	0.3429E-01	-0.1067E 01	-0.1083E 01	0.5707E 00	0.9050E-01
90.	-0.9316E 00	0.1591E 00	0.1987E-01	-0.9515E 00	-0.9495E 00	0.2503E 00	0.9711E-01
ALPHA= 0.15							
0.	0.2085E 01	0.2310E 00	0.2310E 00	0.1854E 01	0.2039E 01	-0.5804E-03	-0.5814E-03
15.	0.2076E 01	0.2169E 00	0.3214E-01	0.2043E 01	0.2051E 01	0.6106E-02	-0.1971E 00
30.	0.1947E 01	-0.4080E 00	0.1783E 00	0.1769E 01	0.1970E 01	-0.6206E 00	0.2438E-01
45.	0.1595E 01	-0.8320E 00	0.1256E 00	0.1469E 01	0.1666E 01	-0.1004E 01	0.4933E-01
60.	-0.1063E 01	0.1050E 01	0.7296E-01	-0.1136E 01	-0.1176E 01	0.1149E 01	0.7429E-01
75.	-0.1049E 01	0.4676E 00	0.3440E-01	-0.1084E 01	-0.1099E 01	0.5691E 00	0.9256E-01
90.	-0.9586E 00	0.1689E 00	0.2028E-01	-0.9789E 00	-0.9775E 00	0.2628E 00	0.9925E-01

BETA = 100.0

ALPHA= 0.95

0.	0.5871E 01	0.6545E 00	0.6545E 00	0.5217E 01	0.5740E 01	0.1918E-02	0.1917E-02
15.	0.5570E 01	0.6069E 00	0.4617E 00	0.5109E 01	0.5463E 01	0.3652E-02	-0.1560E 00
30.	0.4715E 01	0.4767E 00	-0.3185E-01	0.4747E 01	0.4671E 01	0.8391E-02	-0.5510E 00
45.	0.3448E 01	-0.6070E 00	0.2989E 00	0.3149E 01	0.3479E 01	-0.9817E 00	0.1486E-01
60.	0.1994E 01	-0.9958E 00	0.1212E 00	0.1873E 01	0.2082E 01	-0.1207E 01	0.2134E-01
75.	-0.9741E 00	0.6238E 00	-0.8959E-02	-0.9652E 00	-0.1036E 01	0.7221E 00	0.2608E-01
90.	-0.5038E 00	-0.3403E 00	-0.5659E-01	-0.4472E 00	-0.4641E 00	-0.2842E 00	0.2781E-01

ALPHA= 0.75

0.	0.2699E 01	0.2598E 00	0.2598E 00	0.2439E 01	0.2647E 01	-0.3609E-01	-0.3609E-01
15.	0.2618E 01	0.2439E 00	0.1015E 00	0.2517E 01	0.2584E 01	-0.2812E-01	-0.1847E 00
30.	0.2340E 01	-0.2734E 00	0.2003E 00	0.2140E 01	0.2348E 01	-0.5275E 00	-0.6339E-02
45.	0.1832E 01	-0.6576E 00	0.1409E 00	0.1691E 01	0.1884E 01	-0.8549E 00	0.2341E-01
60.	0.1148E 01	-0.8655E 00	0.8145E-01	0.1067E 01	0.1227E 01	-0.9884E 00	0.5317E-01
75.	-0.8027E 00	0.4325E 00	0.3793E-01	-0.8406E 00	-0.8497E 00	0.5090E 00	0.7495E-01
90.	-0.6197E 00	0.2200E-01	0.1055E-01	-0.6303E 00	-0.6230E 00	0.8292E-01	0.7032E-01

ALPHA= 0.55

0.	0.2311E 01	0.2458E 00	0.2458E 00	0.2065E 01	0.2262E 01	-0.9875E-02	-0.9876E-02
15.	0.2282E 01	0.2306E 00	0.5488E-01	0.2227E 01	0.2254E 01	-0.3066E-02	-0.1964E 00
30.	0.2112E 01	-0.3757E 00	0.1891E 00	0.1923E 01	0.2130E 01	-0.6058E 00	0.1554E-01
45.	0.1710E 01	-0.7948E 00	0.1324E 00	0.1577E 01	0.1776E 01	-0.9790E 00	0.4096E-01
60.	0.1110E 01	-0.1016E 01	0.7575E-01	0.1034E 01	0.1204E 01	-0.1135E 01	0.6637E-01
75.	-0.9740E 00	0.4666E 00	0.3424E-01	-0.1008E 01	-0.1024E 01	0.5606E 00	0.8498E-01
90.	-0.8469E 00	0.1195E 00	0.1905E-01	-0.8660E 00	-0.8608E 00	0.2023E 00	0.9179E-01

ALPHA= 0.35

0.	0.2156E 01	0.2381E 00	0.2381E 00	0.1918E 01	0.2108E 01	-0.1266E-02	-0.1267E-02
15.	0.2144E 01	0.2235E 00	0.3679E-01	0.2107E 01	0.2118E 01	0.5355E-02	-0.2000E 00
30.	0.2009E 01	-0.4096E 00	0.1834E 00	0.1825E 01	0.2031E 01	-0.6288E 00	0.2345E-01
45.	0.1644E 01	-0.8394E 00	0.1286E 00	0.1515E 01	0.1715E 01	-0.1017E 01	0.4816E-01
60.	0.1081E 01	-0.1071E 01	0.7386E-01	0.1007E 01	0.1181E 01	-0.1186E 01	0.7288E-01
75.	-0.1049E 01	0.4773E 00	0.3378E-01	-0.1083E 01	-0.1100E 01	0.5788E 00	0.9097E-01
90.	-0.9492E 00	0.1643E 00	0.1910E-01	-0.9683E 00	-0.9676E 00	0.2574E 00	0.9759E-01

ALPHA= 0.15

0.	0.2106E 01	0.2339E 00	0.2339E 00	0.1872E 01	0.2059E 01	-0.7224E-04	-0.7355E-04
15.	0.2097E 01	0.2196E 00	0.3288E-01	0.2064E 01	0.2071E 01	0.6614E-02	-0.1987E 00
30.	0.1967E 01	-0.4124E 00	0.1803E 00	0.1786E 01	0.1990E 01	-0.6271E 00	0.2488E-01
45.	0.1611E 01	-0.8419E 00	0.1267E 00	0.1484E 01	0.1682E 01	-0.1016E 01	0.4984E-01
60.	-0.1077E 01	0.1060E 01	0.7312E-01	-0.1150E 01	-0.1191E 01	0.1161E 01	0.7480E-01
75.	-0.1066E 01	0.4739E 00	0.3388E-01	-0.1100E 01	-0.1117E 01	0.5770E 00	0.9307E-01
90.	-0.9766E 00	0.1742E 00	0.1952E-01	-0.9961E 00	-0.9959E 00	0.2699E 00	0.9976E-01

BETA = 1000.0

ALPHA= 0.95

0.	-0.5859E 00	-0.5859E 00	0.3580E-02	-0.5895E 00	-0.5277E 00	-0.5277E 00	0.1208E 00
15.	-0.5445E 00	-0.5224E 00	0.3342E-02	-0.5478E 00	-0.4926E 00	-0.4683E 00	0.1100E 00
30.	-0.4312E 00	-0.3490E 00	0.2696E-02	-0.4339E 00	-0.3966E 00	-0.3061E 00	0.8072E-01
45.	-0.2765E 00	-0.1121E 00	0.1832E-02	-0.2784E 00	-0.2655E 00	-0.8462E-01	0.4069E-01
60.	0.1249E 00	-0.1219E 00	0.8906E-03	0.1240E 00	0.1370E 00	-0.1344E 00	0.5864E-03
75.	0.2983E 00	-0.8613E-02	0.2576E-03	0.2981E 00	0.2992E 00	-0.3847E-01	-0.2871E-01
90.	0.3618E 00	0.3284E-01	0.2247E-04	0.3618E 00	0.3585E 00	-0.3346E-02	-0.3944E-01

ALPHA= 0.75

0.	-0.3951E 00	-0.3951E 00	0.1362E 00	-0.5313E 00	-0.3692E 00	-0.3692E 00	0.2152E 00
15.	-0.3691E 00	-0.3279E 00	0.1283E 00	-0.4974E 00	-0.3491E 00	-0.3039E 00	0.1980E 00
30.	-0.2980E 00	-0.1553E 00	0.1176E 00	-0.4157E 00	-0.2943E 00	-0.1372E 00	0.1629E 00
45.	0.2084E 00	-0.2010E 00	-0.2482E-01	0.2333E 00	0.2310E 00	-0.2193E 00	-0.2557E-01
60.	0.4381E 00	-0.1039E 00	-0.3321E-01	0.4713E 00	0.4518E 00	-0.1444E 00	-0.6662E-01
75.	0.6240E 00	-0.5713E-01	-0.3287E-01	0.6568E 00	0.6330E 00	-0.1162E 00	-0.8955E-01
90.	0.6930E 00	-0.6693E-01	-0.6865E-02	0.6999E 00	0.7004E 00	-0.1355E 00	-0.6948E-01

ALPHA= 0.55

0.	-0.3427E 00	-0.3427E 00	0.4641E-01	-0.3891E 00	-0.3131E 00	-0.3131E 00	0.1150E 00
15.	-0.3237E 00	-0.2635E 00	0.4278E-01	-0.3665E 00	-0.3016E 00	-0.2354E 00	0.1015E 00
30.	-0.2718E 00	-0.5404E-01	0.3987E-01	-0.3116E 00	-0.2703E 00	-0.3085E-01	0.7245E-01
45.	0.2606E 00	-0.2008E 00	0.7352E-02	0.2533E 00	0.2800E 00	-0.2276E 00	0.1367E-02
60.	0.5568E 00	-0.1298E 00	-0.6632E-02	0.5634E 00	0.5704E 00	-0.1848E 00	-0.4933E-01
75.	0.7747E 00	-0.7784E-01	-0.1799E-01	0.7927E 00	0.7843E 00	-0.1535E 00	-0.8767E-01
90.	0.8545E 00	-0.5882E-01	-0.2221E-01	0.8767E 00	0.8626E 00	-0.1421E 00	-0.1018E 00

ALPHA= 0.35

0.	-0.3471E 00	-0.3471E 00	0.7892E-02	-0.3549E 00	-0.3131E 00	-0.3131E 00	0.7730E-01
15.	-0.3306E 00	-0.2599E 00	0.7187E-02	-0.3378E 00	-0.3054E 00	-0.2276E 00	0.6624E-01
30.	-0.2858E 00	-0.2263E-01	0.5966E-02	-0.2918E 00	-0.2842E 00	0.5355E-02	0.3681E-01
45.	0.3035E 00	-0.2246E 00	0.2326E-02	0.3012E 00	0.3257E 00	-0.2552E 00	-0.5563E-02
60.	0.6287E 00	-0.1634E 00	-0.3433E-03	0.6290E 00	0.6450E 00	-0.2262E 00	-0.4687E-01
75.	0.8667E 00	-0.1186E 00	-0.2317E-02	0.8690E 00	0.8788E 00	-0.2050E 00	-0.7713E-01
90.	0.9539E 00	-0.1022E 00	-0.3041E-02	0.9569E 00	0.9644E 00	-0.1973E 00	-0.8821E-01

ALPHA= 0.15

0.	-0.3486E 00	-0.3486E 00	0.2060E-02	-0.3506E 00	-0.3139E 00	-0.3139E 00	0.7177E-01
15.	-0.3329E 00	-0.2603E 00	0.1911E-02	-0.3348E 00	-0.3071E 00	-0.2272E 00	0.6124E-01
30.	-0.2902E 00	-0.1939E-01	0.1546E-02	-0.2918E 00	-0.2884E 00	0.9478E-02	0.3251E-01
45.	0.3099E 00	-0.2319E 00	0.9382E-03	0.3089E 00	0.3330E 00	-0.2630E 00	-0.6859E-02
60.	0.6391E 00	-0.1736E 00	0.3819E-03	0.6387E 00	0.6564E 00	-0.2375E 00	-0.4617E-01
75.	0.8801E 00	-0.1309E 00	-0.2635E-04	0.8801E 00	0.8932E 00	-0.2189E 00	-0.7495E-01
90.	0.9683E 00	-0.1152E 00	-0.1758E-03	0.9685E 00	0.9798E 00	-0.2120E 00	-0.8548E-01

APPENDIX C

RESULTS FROM TESTS WITH ARTIFICIAL ROCKS

1. Materials and mixtures. Two types of plaster, Hydrocal B-11 and Hydrostone (U. S. Gypsom Co.), were each mixed with water. The true specific gravities of Hydrocal and Hydrostone were found to be 2.19 and 2.30, respectively.

In order to give various porosities, different amounts of water were used and mechanical vibration was sometimes employed. The mix was poured into 2" inside diameter and 12" long cylindrical plastic tubes. The mix was hardened enough to be taken out of the mold in about two to five hours, depending on the mixture. Specimens thus made were air-dried and cured at room temperature for about thirty days, before they were ready to be tested.

2. Tests made. All the strength tests were made with a Tinus-Olsen testing machine of 120,000 lbs. capacity.

a. Flexural strength test. The specimens were tested without cutting, in a center-point loading device with a span of 5". The edge of the upper plate was carefully lined with the center of the lower plate to minimize the effect of non-uniform shear in the specimen. All specimens were loaded at a rate of 400 lbs/min.

b. Uniaxial compressive strength test. About 4" from the top and bottom of 12" specimens were cut off with a diamond saw to obtain homogeneous specimens. The length of the specimen was about twice the diameter. The top and bottom faces were made smooth and parallel to each other with a grinding machine within ±.005". The specimen was then placed in the testing machine and loaded at a rate of 60 psi/sec.

c. Uniaxial tensile strength test. The preparation of the specimen was the same as for the compressive test. The specimen was glued to the upper and lower platens with structural adhesive. The platens are connected to the loading plates with a pair of roller chains to prevent moment from developing in the specimen. The plates attached on upper and lower loading plates to hold the chains were designed so that the center line of specimen will lie within ±0.020" from that of the loading machine (65). When the specimen broke at the ends very close to the platens, the results were discarded. The loading rate was 50 psi/sec.

d. Brazilian (indirect tensile) test. Specimens for this test were cut off from the flexural test specimens after they failed. The lengths vary from 1" to 2". The loading rate was about 400 to 500 lbs/min., depending on the specimens.

e. Apparent porosity. All specimens tested were measured for the apparent porosities. The dry and wet weights of specimens were obtained by weighing specimens before and after immersing them in water for twenty four hours.

f. Young's modulus. Wire type polyester strain gages (Tokyo Sokki Co.), ½ cm. size, were attached to each sample with Eastmann 910 adhesive. All the values given here are initial moduli.

The test results are listed on the following pages. The number of specimens tested for each mix is four unless indicated otherwise in the parenthesis after each value. The strengths shown are the averages of them. The symbols Sf, Sc, St, and Sb denote flexural, compressive, tensile, and indirect tensile strengths, respectively.

TABLE I. TESTS WITH HYDROCAL B-11

No.	Porosity %	E 10 psi	Sf psi	Sc psi	St psi	Sb psi
1	30.5	1.12	524	1291	188	201
2	52.5	.25 (2)	274 (6)	333 (5)	85 (5)	75 (12)
3	42.5	.40 (2)	342 (6)	642 (6)	150 (5)	140 (12)
4	18.0	1.57	920	1875	410	404
5	3.0	2.25	1494	4896	603	551
6	32.5	1.06	579	1318	307	234
7	36.0	.93	432	822	142	184
8	29.0	1.20	627	1692	336	265

TABLE II. TESTS WITH HYDROSTONES

No.	Porosity %	E 10 psi	Sf psi	Sc psi	St psi	Sb psi
1	54.5	.25 (3)	124 (3)	154 (3)	51 (3)	150 (3)
2	41.0	.50 (3)	235 (3)	805 (3)	220 (3)	238
3	19.5	1.40	991	2381	364	432
4	18.0	1.60	1071	3196	504	449
5	30.0	1.15	741	1218	240	244 (6)
6	27.0	1.18 (2)	715 (2)	1532 (2)	258 (2)	239
7	5.5	2.10	1384	3417	537	524
8	10.5	2.00	1324	2300	430	462

APPENDIX D

CONSTANTS FOR STRESS SOLUTIONS IN MATRIX REGION

BASED ON $E_1 = 10^5$ psi and $\nu_1 = 0.1$

$\beta = .001$

γ	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	-0.7905E-04	0.6410E-04	-0.1990E-03	0.3806E-03	0.1535E-04	0.9290E-04
	.75	-0.4834E-04	0.2535E-04	-0.1598E-03	0.2868E-03	0.5047E-04	0.5827E-04
	.55	-0.2380E-04	0.6583E-05	-0.9361E-04	0.1675E-03	0.4110E-04	0.2870E-04
	.35	-0.7419E-05	0.8198E-06	-0.2946E-04	0.4204E-04	0.1613E-04	0.8910E-05
	.15	-0.6158E-06	0.1274E-07	-0.2427E-05	0.2608E-05	0.1525E-05	0.7549E-06
$\gamma = .6$	.95	-0.8892E-04	0.6661E-04	-0.1924E-03	0.3323E-03	0.1595E-04	0.9655E-04
	.75	-0.5411E-04	0.2630E-04	-0.1520E-03	0.2295E-03	0.5238E-04	0.6047E-04
	.55	-0.2557E-04	0.6746E-05	-0.8688E-04	0.1230E-03	0.4211E-04	0.2941E-04
	.35	-0.7521E-05	0.8239E-06	-0.2779E-04	0.3198E-04	0.1621E-04	0.8955E-05
	.15	-0.6160E-06	0.1274E-07	-0.2359E-05	0.2190E-05	0.1523E-05	0.7551E-06
$\gamma = 1.$	.95	-0.9881E-04	0.6914E-04	-0.1859E-03	0.2839E-03	0.1655E-04	0.1002E-03
	.75	-0.5989E-04	0.2726E-04	-0.1442E-03	0.1720E-03	0.5429E-04	0.6267E-04
	.55	-0.2733E-04	0.6909E-05	-0.8014E-04	0.7846E-04	0.4313E-04	0.3012E-04
	.35	-0.7623E-05	0.8281E-06	-0.2611E-04	0.2189E-04	0.1629E-04	0.9000E-05
	.15	-0.6163E-06	0.1275E-07	-0.2289E-05	0.1771E-05	0.1534E-05	0.7553E-06
$\gamma = 4.$	.95	-0.1723E-03	0.8800E-04	-0.1366E-03	-0.7950E-04	0.2107E-04	0.1275E-03
	.75	-0.1028E-03	0.3444E-04	-0.8577E-04	-0.2590E-03	0.6859E-04	0.7917E-04
	.55	-0.4039E-04	0.8128E-05	-0.2970E-04	-0.2554E-03	0.5075E-04	0.3544E-04
	.35	-0.8354E-05	0.8589E-06	-0.1358E-04	-0.5363E-04	0.1689E-04	0.9335E-05
	.15	-0.6157E-06	0.1277E-07	-0.1752E-05	-0.1373E-05	0.1533E-05	0.7567E-06

$\beta = .10$	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	-0.7215E-04	0.5668E-04	-0.1795E-03	0.3608E-03	0.1387E-04	0.8229E-04
	.75	-0.4325E-04	0.2170E-04	-0.1410E-03	0.2722E-03	0.4356E-04	0.5006E-04
	.55	-0.2095E-04	0.5481E-05	-0.8091E-04	0.1593E-03	0.3427E-04	0.2393E-04
	.35	-0.6456E-05	0.6656E-06	-0.2460E-04	0.3929E-04	0.1310E-04	0.7236E-05
	.15	-0.5348E-06	0.1022E-07	-0.1986E-05	0.2375E-05	0.1235E-05	0.6054E-06
$\gamma = .6$	.95	-0.8046E-04	0.5918E-04	-0.1730E-03	0.3080E-03	0.1451E-04	0.8594E-04
	.75	-0.4791E-04	0.2265E-04	-0.1333E-03	0.2117E-03	0.4549E-04	0.5226E-04
	.55	-0.2224E-04	0.5655E-05	-0.7433E-04	0.1135E-03	0.3537E-04	0.2469E-04
	.35	-0.6471E-05	0.6743E-06	-0.2307E-04	0.2896E-04	0.1326E-04	0.7330E-05
	.15	-0.5281E-06	0.1031E-07	-0.1931E-05	0.1937E-05	0.1244E-05	0.6107E-06
$\gamma = 1.$	.95	-0.8848E-04	0.6169E-04	-0.1664E-03	0.2542E-03	0.1514E-04	0.8959E-04
	.75	-0.5236E-04	0.2360E-04	-0.1255E-03	0.1503E-03	0.4742E-04	0.5446E-04
	.55	-0.2343E-04	0.5830E-05	-0.6773E-04	0.6724E-04	0.3646E-04	0.2545E-04
	.35	-0.6459E-05	0.6827E-06	-0.2153E-04	0.1854E-04	0.1343E-04	0.7421E-05
	.15	-0.5200E-06	0.1039E-07	-0.1852E-05	0.1493E-05	0.1241E-05	0.6157E-06
$\gamma = 4.$	.95	-0.1013E-03	0.8036E-04	-0.1174E-03	-0.2836E-03	0.2014E-04	0.1169E-03
	.75	-0.5519E-04	0.3065E-04	-0.6720E-04	-0.3958E-03	0.6203E-04	0.7095E-04
	.55	-0.2022E-04	0.7107E-05	-0.1809E-04	-0.3134E-03	0.4453E-04	0.3106E-04
	.35	-0.4018E-05	0.7381E-06	-0.9778E-05	-0.6610E-04	0.1452E-04	0.8025E-05
	.15	-0.2921E-06	0.1088E-07	-0.1408E-05	-0.2303E-05	0.1308E-05	0.6445E-06

$\beta = 1.001$	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	-0.1248E-04	-0.4631E-05	-0.1385E-04	0.1892E-03	-0.1349E-05	-0.6829E-05
	.75	-0.6384E-05	-0.1403E-05	-0.1801E-04	0.1662E-03	-0.2984E-05	-0.3318E-05
	.55	-0.2766E-05	-0.3014E-06	-0.1369E-04	0.1070E-03	-0.1911E-05	-0.1327E-05
	.35	-0.8036E-06	-0.3245E-07	-0.2578E-05	0.2302E-04	-0.6432E-06	-0.3531E-06
	.15	-0.6512E-07	-0.4726E-09	0.5484E-07	0.1026E-05	-0.9498E-07	-0.2800E-07
$\gamma = .6$	.95	-0.7585E-05	-0.2351E-05	-0.6820E-05	0.9846E-04	-0.6926E-06	-0.3471E-05
	.75	-0.3808E-05	-0.7169E-06	-0.8913E-05	0.8488E-04	-0.1529E-05	-0.1697E-05
	.55	-0.1570E-05	-0.1530E-06	-0.6811E-05	0.5405E-04	-0.9701E-06	-0.6739E-06
	.35	-0.4286E-06	-0.1621E-07	-0.1309E-05	0.1159E-04	-0.3040E-06	-0.1764E-06
	.15	-0.3423E-07	-0.2352E-09	0.2693E-08	0.5179E-06	-0.9989E-07	-0.1394E-07
$\gamma = 1.$	.95	0.1164E-06	-0.5727E-07	0.1884E-06	-0.5251E-06	-0.1333E-07	-0.8461E-07
	.75	0.5726E-07	-0.1756E-07	0.1292E-06	-0.3490E-06	-0.3793E-07	-0.4163E-07
	.55	0.2258E-07	-0.3724E-08	0.6413E-07	-0.1883E-06	-0.2250E-07	-0.1641E-07
	.35	0.5817E-08	-0.3886E-09	0.1397E-07	-0.4201E-07	-0.4776E-08	-0.4229E-08
	.15	0.4579E-09	-0.5617E-11	0.3487E-08	-0.2058E-08	-0.4546E-08	-0.3329E-09
$\gamma = 4.$	.95	0.4169E-03	0.1713E-04	0.5127E-04	-0.1773E-02	0.5884E-05	0.2570E-04
	.75	0.1389E-03	0.5427E-05	0.6578E-04	-0.9539E-03	0.1211E-04	0.1311E-04
	.55	0.3990E-04	0.1119E-05	0.5137E-04	-0.4862E-03	0.7161E-05	0.4964E-05
	.35	0.7097E-05	0.1056E-06	0.1017E-04	-0.9805E-04	0.2079E-05	0.1150E-05
	.15	0.4983E-06	0.1487E-08	0.2798E-06	-0.4576E-05	0.1708E-06	0.8810E-07

$\beta = 3.0$	α	A_1	B_1	C_1	D_1	E_1	F_1
	.95	0.1028E-03	-0.1099E-03	0.2923E-03	-0.1423E-03	-0.4204E-04	-0.1670E-03
	.75	0.4100E-04	-0.2438E-04	0.1143E-03	0.2997E-04	-0.5598E-04	-0.5965E-04
$\gamma = .2$	.55	0.1564E-04	-0.4562E-05	0.3700E-04	0.5410E-04	-0.2935E-04	-0.2031E-04
	.35	0.4289E-05	-0.4546E-06	0.1077E-04	0.8374E-05	-0.8964E-05	-0.4955E-05
	.15	0.3414E-06	-0.6423E-08	0.1006E-05	-0.1433E-06	-0.7713E-06	-0.3806E-06
	.95	0.1316E-03	-0.1086E-03	0.3027E-03	-0.3017E-03	-0.4222E-04	-0.1653E-03
	.75	0.5085E-04	-0.2437E-04	0.1275E-03	-0.7226E-04	-0.5620E-04	-0.5974E-04
$\gamma = .6$	.55	0.1841E-04	-0.4533E-05	0.4531E-04	-0.3382E-05	-0.2919E-04	-0.2020E-04
	.35	0.4743E-05	-0.4445E-06	0.1226E-04	-0.3282E-05	-0.8767E-05	-0.4845E-05
	.15	0.3721E-06	-0.6257E-08	0.1057E-05	-0.6497E-06	-0.7492E-06	-0.3708E-06
	.95	0.1670E-03	-0.1072E-03	0.3131E-03	-0.4803E-03	-0.4243E-04	-0.1635E-03
	.75	0.6234E-04	-0.2433E-04	0.1405E-03	-0.1794E-03	-0.5635E-04	-0.5976E-04
$\gamma = 1.$	.55	0.2154E-04	-0.4501E-05	0.5361E-04	-0.6204E-04	-0.2901E-04	-0.2007E-04
	.35	0.5235E-05	-0.4345E-06	0.1375E-04	-0.1508E-04	-0.8571E-05	-0.4736E-05
	.15	0.4048E-06	-0.6092E-08	0.1100E-05	-0.1164E-05	-0.7286E-06	-0.3610E-06
	.95	0.1094E-02	-0.9334E-04	0.3872E-03	-0.3720E-02	-0.4675E-04	-0.1471E-03
	.75	0.2419E-03	-0.2292E-04	0.2331E-03	-0.1250E-02	-0.5601E-04	-0.5772E-04
$\gamma = 4.$	.55	0.6233E-04	-0.4162E-05	0.1146E-03	-0.5507E-03	-0.2711E-04	-0.1869E-04
	.35	0.1067E-04	-0.3627E-06	0.2499E-04	-0.1083E-03	-0.7163E-05	-0.3958E-05
	.15	0.7409E-06	-0.4948E-08	0.1440E-05	-0.5274E-05	-0.5999E-06	-0.2932E-06

$\beta = 10.$

γ	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	0.3832E-03	-0.3119E-03	0.9806E-03	-0.9484E-03	-0.1882E-03	-0.5072E-03
	.75	0.9947E-04	-0.4527E-04	0.2502E-03	-0.1381E-03	-0.1147E-03	-0.1160E-03
	.55	0.3307E-04	-0.7594E-05	0.7443E-04	0.4003E-05	-0.4980E-04	-0.3427E-04
	.35	0.8611E-05	-0.7214E-06	0.1925E-04	-0.4053E-05	-0.1426E-04	-0.7875E-05
	.15	0.6753E-06	-0.1001E-07	0.1676E-05	-0.1105E-05	-0.1208E-05	-0.5934E-06
$\gamma = .6$	.95	0.4614E-03	-0.3150E-03	0.1009E-02	-0.1250E-02	-0.1938E-03	-0.5139E-03
	.75	0.1153E-03	-0.4634E-04	0.2702E-03	-0.2576E-03	-0.1178E-03	-0.1189E-03
	.55	0.3650E-04	-0.7710E-05	0.8449E-04	-0.5540E-04	-0.5059E-04	-0.3481E-04
	.35	0.8948E-05	-0.7195E-06	0.2100E-04	-0.1537E-04	-0.1422E-04	-0.7855E-05
	.15	0.6919E-06	-0.9943E-08	0.1715E-05	-0.1570E-05	-0.1188E-05	-0.5892E-06
$\gamma = 1.$	.95	0.5507E-03	-0.3176E-03	0.1038E-02	-0.1583E-02	-0.1997E-03	-0.5203E-03
	.75	0.1323E-03	-0.4738E-04	0.2901E-03	-0.3804E-03	-0.1209E-03	-0.1218E-03
	.55	0.4011E-04	-0.7823E-05	0.9455E-04	-0.1154E-03	-0.5137E-04	-0.3534E-04
	.35	0.9297E-05	-0.7175E-06	0.2274E-04	-0.2676E-04	-0.1418E-04	-0.7834E-05
	.15	0.7091E-06	-0.9874E-08	0.1777E-05	-0.2041E-05	-0.1174E-05	-0.5851E-06
$\gamma = 4.$	.95	0.1946E-02	-0.3245E-03	0.1241E-02	-0.6170E-02	-0.2581E-03	-0.5579E-03
	.75	0.3024E-03	-0.5390E-04	0.4341E-03	-0.1424E-02	-0.1427E-03	-0.1411E-03
	.55	0.7347E-04	-0.8577E-05	0.1688E-03	-0.5827E-03	-0.5669E-04	-0.3892E-04
	.35	0.1234E-04	-0.7025E-06	0.3578E-04	-0.1131E-03	-0.1390E-04	-0.7675E-05
	.15	0.8529E-06	-0.9369E-08	0.2233E-05	-0.5595E-05	-0.1117E-05	-0.5553E-06

$\beta = 30.$	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	0.7469E-03	-0.4999E-03	0.1800E-02	-0.1994E-02	-0.4450E-03	-0.8823E-03
	.75	0.1335E-03	-0.5457E-04	0.3195E-03	-0.2358E-03	-0.1467E-03	-0.1439E-03
	.55	0.4128E-04	-0.8764E-05	0.8948E-04	-0.1961E-04	-0.5809E-04	-0.3985E-04
	.35	0.1050E-04	-0.8190E-06	0.2237E-04	-0.9484E-05	-0.1621E-04	-0.8950E-05
	.15	0.8181E-06	-0.1130E-07	0.1887E-05	-0.1513E-05	-0.1352E-05	-0.6696E-06
$\gamma = .6$	.95	0.8717E-03	-0.5114E-03	0.1865E-02	-0.2430E-02	-0.4617E-03	-0.9057E-03
	.75	0.1514E-03	-0.5634E-04	0.3439E-03	-0.3613E-03	-0.1517E-03	-0.1487E-03
	.55	0.4477E-04	-0.8952E-05	0.1004E-03	-0.7917E-04	-0.5935E-04	-0.4071E-04
	.35	0.1074E-04	-0.8210E-06	0.2424E-04	-0.2051E-04	-0.1625E-04	-0.8972E-05
	.15	0.8252E-06	-0.1127E-07	0.1952E-05	-0.1954E-05	-0.1349E-05	-0.6681E-06
$\gamma = 1.$	.95	0.1006E-02	-0.5224E-03	0.1930E-02	-0.2894E-02	-0.4788E-03	-0.9286E-03
	.75	0.1698E-03	-0.5809E-04	0.3683E-03	-0.4883E-03	-0.1567E-03	-0.1535E-03
	.55	0.4832E-04	-0.9138E-05	0.1113E-03	-0.1391E-03	-0.6061E-04	-0.4157E-04
	.35	0.1098E-04	-0.8230E-06	0.2611E-04	-0.3158E-04	-0.1628E-04	-0.8994E-05
	.15	0.8324E-06	-0.1125E-07	0.2027E-05	-0.2394E-05	-0.1347E-05	-0.6667E-06
$\gamma = 4.$	.95	0.2451E-02	-0.5877E-03	0.2395E-02	-0.7621E-02	-0.6222E-03	-0.1085E-02
	.75	0.3238E-03	-0.7050E-04	0.5482E-03	-0.1485E-02	-0.1935E-03	-0.1878E-03
	.55	0.7710E-04	-0.1049E-04	0.1927E-03	-0.5932E-03	-0.6978E-04	-0.4782E-04
	.35	0.1288E-04	-0.8372E-06	0.4006E-04	-0.1147E-03	-0.1657E-04	-0.9151E-05
	.15	0.8884E-06	-0.1107E-07	0.2545E-05	-0.5697E-05	-0.1329E-05	-0.6560E-06

$\beta = 100.$	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	0.1061E-02	-0.6209E-03	0.2471E-02	-0.2896E-02	-0.7066E-03	-0.1170E-02
	.75	0.1494E-03	-0.5835E-04	0.3500E-03	-0.2816E-03	-0.1614E-03	-0.1560E-03
	.55	0.4474E-04	-0.9215E-05	0.9542E-04	-0.2957E-04	-0.6138E-04	-0.4205E-04
	.35	0.1127E-04	-0.8561E-06	0.2356E-04	-0.1170E-04	-0.1695E-04	-0.9359E-05
	.15	0.8759E-06	-0.1178E-07	0.1969E-05	-0.1680E-05	-0.1410E-05	-0.6983E-06
$\gamma = .6$	.95	0.1211E-02	-0.6412E-03	0.2573E-02	-0.3405E-02	-0.7343E-03	-0.1211E-02
	.75	0.1679E-03	-0.6045E-04	0.3766E-03	-0.4088E-03	-0.1673E-03	-0.1617E-03
	.55	0.4820E-04	-0.9433E-05	0.1067E-03	-0.8905E-04	-0.6284E-04	-0.4305E-04
	.35	0.1146E-04	-0.8597E-06	0.2548E-04	-0.2259E-04	-0.1702E-04	-0.9399E-05
	.15	0.8785E-06	-0.1178E-07	0.2042E-05	-0.2107E-05	-0.1414E-05	-0.6979E-06
$\gamma = 1.$	.95	0.1366E-02	-0.6612E-03	0.2675E-02	-0.3928E-02	-0.7623E-03	-0.1251E-02
	.75	0.1866E-03	-0.6255E-04	0.4032E-03	-0.5368E-03	-0.1732E-03	-0.1674E-03
	.55	0.5169E-04	-0.9651E-05	0.1180E-03	-0.1487E-03	-0.6430E-04	-0.4404E-04
	.35	0.1164E-04	-0.8633E-06	0.2740E-04	-0.3350E-04	-0.1709E-04	-0.9438E-05
	.15	0.8811E-06	-0.1177E-07	0.2119E-05	-0.2535E-05	-0.1416E-05	-0.6976E-06
$\gamma = 4.$	.95	0.2689E-02	-0.8011E-03	0.3422E-02	-0.8306E-02	-0.9787E-03	-0.1543E-02
	.75	0.3318E-03	-0.7799E-04	0.6010E-03	-0.1508E-02	-0.2172E-03	-0.2093E-03
	.55	0.7843E-04	-0.1127E-04	0.2025E-03	-0.5970E-03	-0.7514E-04	-0.5145E-04
	.35	0.1307E-04	-0.8900E-06	0.4174E-04	-0.1152E-03	-0.1762E-04	-0.9730E-05
	.15	0.9012E-06	-0.1173E-07	0.2656E-05	-0.5734E-05	-0.1404E-05	-0.6949E-06

$\beta = 1000.$	α	A_1	B_1	C_1	D_1	E_1	F_1
$\gamma = .2$	.95	0.1257E-02	-0.6833E-03	0.2879E-02	-0.3459E-02	-0.8834E-03	-0.1339E-02
	.75	0.1563E-03	-0.5988E-04	0.3630E-03	-0.3014E-03	-0.1677E-03	-0.1611E-03
	.55	0.4618E-04	-0.9395E-05	0.9781E-04	-0.3370E-04	-0.6271E-04	-0.4293E-04
	.35	0.1159E-04	-0.8708E-06	0.2404E-04	-0.1261E-04	-0.1724E-04	-0.9522E-05
	.15	0.8994E-06	-0.1198E-07	0.2003E-05	-0.1748E-05	-0.1431E-05	-0.7097E-06
$\gamma = .6$	.95	0.1416E-02	-0.7096E-03	0.3005E-02	-0.3994E-02	-0.9180E-03	-0.1391E-02
	.75	0.1750E-03	-0.6213E-04	0.3905E-03	-0.4292E-03	-0.1740E-03	-0.1672E-03
	.55	0.4962E-04	-0.9626E-05	0.1093E-03	-0.9313E-04	-0.6426E-04	-0.4399E-04
	.35	0.1175E-04	-0.8751E-06	0.2597E-04	-0.2343E-04	-0.1733E-04	-0.9569E-05
	.15	0.9001E-06	-0.1198E-07	0.2074E-05	-0.2170E-05	-0.1429E-05	-0.7098E-06
$\gamma = 1.$	.95	0.1576E-02	-0.7359E-03	0.3132E-02	-0.4532E-02	-0.9528E-03	-0.1443E-02
	.75	0.1938E-03	-0.6438E-04	0.4180E-03	-0.5574E-03	-0.1803E-03	-0.1733E-03
	.55	0.5307E-04	-0.9857E-05	0.1207E-03	-0.1527E-03	-0.6580E-04	-0.4505E-04
	.35	0.1192E-04	-0.8794E-06	0.2791E-04	-0.3428E-04	-0.1741E-04	-0.9615E-05
	.15	0.9008E-06	-0.1198E-07	0.2154E-05	-0.2591E-05	-0.1435E-05	-0.7099E-06
$\gamma = 4.$	.95	0.2793E-02	-0.9312E-03	0.4075E-02	-0.8605E-02	-0.1214E-02	-0.1830E-02
	.75	0.3350E-03	-0.8120E-04	0.6238E-03	-0.1518E-02	-0.2276E-03	-0.2186E-03
	.55	0.7895E-04	-0.1158E-04	0.2065E-03	-0.5985E-03	-0.7734E-04	-0.5294E-04
	.35	0.1315E-04	-0.9112E-06	0.4242E-04	-0.1154E-03	-0.1804E-04	-0.9964E-05
	.15	0.9062E-06	-0.1199E-07	0.2701E-05	-0.5748E-05	-0.1433E-05	-0.7106E-06

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